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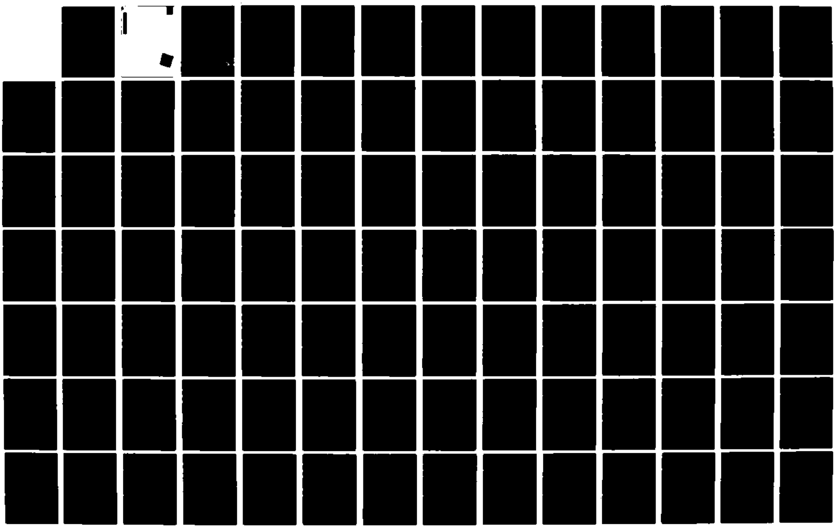

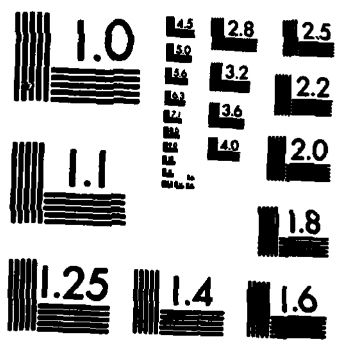

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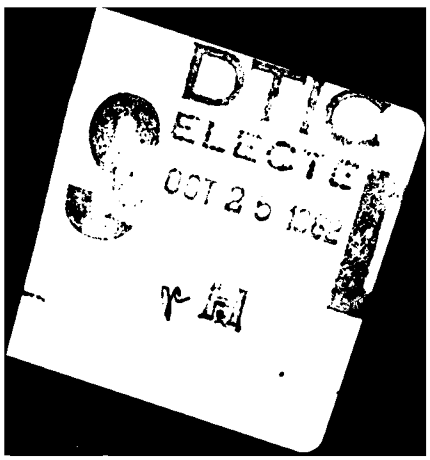
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A PRODUCTION NETWORK MODEL AND ITS DIFFUSION APPROXIMATION

by

Michael Louis Wenocur

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Abstract

This report develops and analyzes a general stochastic model of a production system. The model is closely related to Harrison's [5] assembly-like queueing network, the principal difference being that here we assume all storage buffers have finite capacity. Our attention is focused on a vector stochastic process Z whose components are the contents of the various storage buffers (as functions of time). The principal result is a weak convergence theorem of the type developed by Iglehart and Whitt [7] for queues in heavy traffic. This limit theorem shows that, with large buffers and balanced loading of the system's work stations (see below), a properly normalized version of the storage process Z can be well approximated by a certain vector diffusion process Z^*. We construct Z^* by applying a particular (and rather complicated) reflection mapping to multidimensional Brownian motion. Various properties of the limiting diffusion Z^* are developed, but these provide only a modest beginning for the analytical theory that must be developed before our limit theorem can lead to practically useful approximation procedures.

KEY WORDS

QUEUES	REFLECTED BROWNIAN MOTION
BUFFERS	REFLECTION MAPPING
	DIFFUSION APPROXIMATION

CHAPTER 1

INTRODUCTION AND SUMMARY

This report develops and analyzes a general stochastic model of a production system. The model is closely related to Harrison's [5] assembly-like queueing network, the principal difference being that here we assume all storage buffers have finite capacity. Our attention is focused on a vector stochastic process Z whose components are the contents of the various storage buffers (as functions of time). The principal result is a weak convergence theorem of the type developed by Iglehart and Whitt [7] for queues in heavy traffic. This limit theorem shows that, with large buffers and balanced loading of the system's work stations (see below), a properly normalized version of the storage process Z can be well approximated by a certain vector diffusion process Z^*. We construct Z^* by applying a particular (and rather complicated) reflection mapping to multidimensional Brownian motion. Various properties of the limiting diffusion Z^* are developed, but these provide only a modest beginning for the analytical theory that must be developed before our limit theorem can lead to practically useful approximation procedures.

1.1 The Systems Being Modelled

A simple example of the systems under study is the assembly operation pictured in Figure 1. Input items of types 1 and 2 are generated by external sources and deposited into similarly numbered storage buffers. We call these external sources work stations 1 and

2. Assemblers at work station 3 then combine the input items into finished goods (type 3 items). Let us assume that exactly one item each of types 1 and 2 is required to make an item of type 3, and type 3 items depart the system immediately upon completion. Work at station 1 or 2 must stop if the corresponding storage buffer is full, and work at station 3 must stop if either of the two storage buffers is empty. In the former case, potential production from the input station is lost because of what we will call <u>blockage</u>. In the latter case, potential production from the assembly station is lost because of <u>starvation</u>.

In the description above, we have talked in terms of discrete items, but Figure 1 could also represent a blending operation in which granulated or liquid ingredients are combined in fixed proportions to produce a similarly continuous output product. In that case, one would speak of external sources delivering <u>material</u> of types 1 and 2 which is blended to produce output <u>material</u> of type 3, and the storage buffers might be called <u>surge tanks</u> in the case of liquid flows. Even when speaking of continuous flow systems, we will nonetheless employ the language of work stations and storage buffers.

A more complex sort of system is the production <u>network</u> pictured in Figure 2. Here we have external work stations supplying material of types 1, ..., 5, plus a succession of internal work stations that transform these inputs by stages into an output material of type 10. Work stations 6, 8 and 10 have multiple inputs, so they involve some sort of assembly or blending. Internal stations with a single input might represent such transformations as the stamping of blanks from

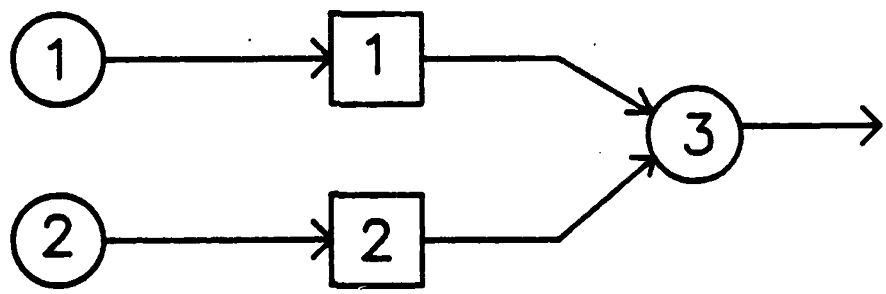

Figure 1. An Assembly or Blending Operation. Circles Represent Work Stations and Squares Represent Storage Buffers.

sheet metal, or the cooling of hot liquid input in a heat exchanger. Note that each work station in Figure 2, except the last, deposits its output material into a buffer of finite capacity. The work station, output material and storage buffer are all designated by the same number. Attention will be restricted here to production networks in which all assembly or blending operations use inputs in fixed proportions, each work station produces a single type of material as output, and each type of material (except the finished product) is used as input at a single downstream station.

Typically, the amount of output that a work station can produce in any given period is stochastically variable, due to worker absenteeism, mechanical failures, quality variations in raw materials, and so forth. It is this stochastic variability that leads to non-zero inventory levels in the storage buffers and to lost potential output due to blockage and starvation. Our objective in modeling is to understand how system performance characteristics, like average inventory levels and average throughput rates, depend on distributional properties of the work rates at various stations.

1.2 Stochastic Models Employed

Our ultimate purpose is to propose a class of diffusion processes as models of production networks. It is not easy to see that these processes are appropriate for that task, or to see the conditions under which they are appropriate. Thus, as an aid to intuition, it will be shown that the diffusion model represents the limit of more readily comprehensible models of conventional type. With regard to

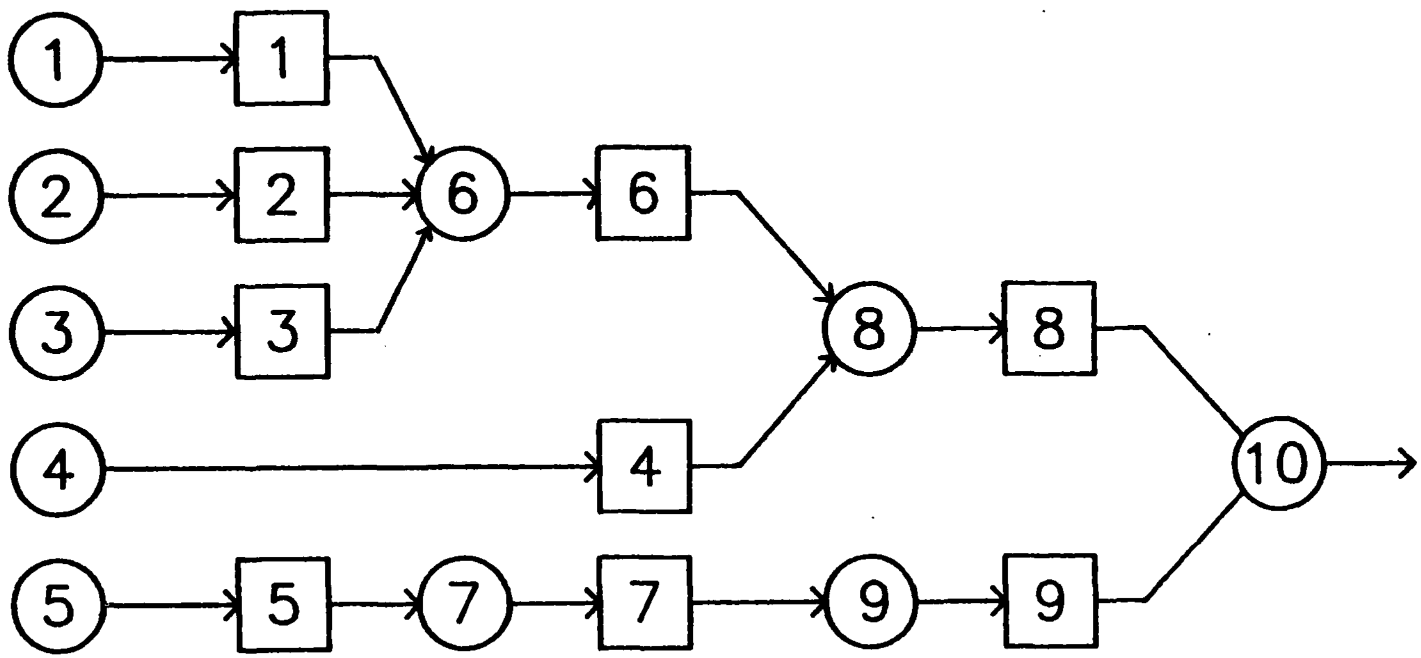

Figure 2. A Production Network. Circles Represent Work Stations and Squares Represent Storage Buffers.

the precise characteristics of the so-called conventional model, we have quite a bit of latitude. The particular conventional model to be discussed here was chosen with some very specific criteria in mind. We will now describe it, restricting attention to the simple assembly operation of Figure 1 for simplicity, and will afterward discuss its weaknesses and virtues.

To model the assembly or blending operation of Figure 1, we take as primitive three increasing processes $\xi_k = (\xi_k(t),\ t \geq 0)$ satisfying $\xi_k(0) = 0$ $(k = 1,2,3)$. Call ξ_k the <u>potential output process</u> for work station k, interpreting $\xi_k(t) - \xi_k(s)$ as the total output (total amount of material k) that station k can produce over the time interval $[s,t]$ if it is able to work without interruption during that period. If blockage (in the case of input stations 1 and 2) or starvation (in the case of blending station 3) occurs, then the actual output will be less (see below). Define $X_k(t) \equiv \xi_k(t)-\xi_3(t)$ for $k = 1,2$ and $t \geq 0$. Denoting by b_k the capacity of buffer k $(0 < b_k < \infty)$, we assume as given initial contents $Z_k(0)$ such that $0 \leq Z_k(0) \leq b_k$ $(k = 1,2)$. A principle modeling task is to explain how the <u>contents process</u> $Z(t) = (Z_1(t),\ Z_2(t))$ is defined in terms of the primitive model elements for $t > 0$. If there is no blockage or starvation up to time t, then we have simply

$$(1) \qquad Z_k(t) = Z_k(0) + \xi_k(t) - \xi_3(t) = Z_k(0) + X_k(t)$$

for $t \geq 0$. Thus $X(t) \equiv (X_1(t),\ X_2(t))$ is called the <u>net potential</u>

input process underlying our system model. How can (1) be modified to account for potential blockage of the input stations and/or starvation of the blending station? Let $Y_k(t)$ denote the lost potential output from station k (due to blockage or starvation) up to time t, so that actual output from station k over the interval $[0,t]$ is $\xi_k(t) - Y_k(t)$. Then the correct modification of (1) is

$$
\begin{aligned}
(2) \qquad Z_k(t) &= Z_k(0) + (\xi_k(t) - Y_k(t)) - (\xi_3(t) - Y_3(t)) \\
&= Z_k(0) + X_k(t) - Y_k(t) + Y_3(t) \quad .
\end{aligned}
$$

for $k = 1,2$ and $t \geq 0$. But now we obviously have the problem of defining $Y(t) \equiv (Y_1(t), Y_2(t), Y_3(t))$ precisely in terms of primitive model elements.

Before going further, we introduce the critical final assumption that ξ_k is a continuous process for each $k = 1,2,3$. If Figure 1 is interpreted as a blending operation, then this assumption is non-controversial, but if it represents an assembly operation for manufactured items, the continuity assumption constitutes a potentially gross approximation of reality. In the latter case, our defense is that we only seek to analyze well balanced high volume systems (see Chapter 6) with relatively large buffers. One may then reasonably approximate the cumulative output from a work station by a continuous function of time, and the content of a buffer may be viewed as an approximately continuous variable. Be that as it may, with ξ_k assumed continuous, it is reasonable to require that

(3) Y_k is continuous and increasing with $Y_k(0) = 0$

for each $k = 1,2,3$. Then Z will be a continuous process as well by virtue of (2). Finally, the intended meanings of Y and Z suggest the key relations

(4) $$0 \leq Z_k(t) \leq b_k \qquad \text{for} \quad k = 1,\ 2 \quad \text{and} \quad t \geq 0,$$

(5) $$\int_0^t (b_k - Z_k(s))\, dY_k(s) = 0 \qquad \text{for} \quad k = 1,\ 2 \quad \text{and} \quad t \geq 0,$$

(6) $$\int_0^t (Z_1(s) \wedge Z_2(s))\, dY_3(s) = 0 \quad \text{for} \quad t \geq 0\ .$$

The meaning of (4) is clear, and (5) says that potential output from station $k = 1,2$ is lost (Y_k increases) only when $Z_k = b_k$. Equivalently, potential output from station $k = 1,2$ is foregone or sacrificed in the minimum amounts necessary to maintain $Z_k(t) \leq b_k$ for $t \geq 0$. In precisely parallel fashion, (6) says that potential output from the blending operation is lost only when one or both of the storage buffers is empty, and together with (4) this means that potential blending output is sacrificed in the minimum amounts necessary to insure that $Z_k(t) \geq 0$ for all $t \geq 0$ and $k = 1,2$.

We have thus far dodged the question of how one defines Y precisely in terms of ξ and $Z(0)$, while simply listing (3)-(6) as necessary properties. It turns out, however, that these properties uniquely determine Y. The following is a special case of a representation theorem for general networks (like that pictured in

Figure 2) to be proved in Chapter 4. Let S (for state space) denote the rectangle $[0,b_1] \times [0,b_2]$.

(7) Theorem. For each continuous X and $Z(0) \in S$, there exists a unique Y satisfying (3)-(6), where Z is defined in terms of Y by (2).

There remains the essential task of specifying the stochastic character of our potential output process ξ. A number of different assumptions will ultimately be considered, but let us focus on the following for the sake of concreteness. Assume that there is given an IID sequence of positive random three-vectors

$$\zeta = \{(\zeta_1(n), \zeta_2(n), \zeta_3(n)); n = 1,2,3...\} .$$

We now define ξ in terms of ζ by means of

(8) $\xi_k(n) \equiv \zeta_k(1) + \zeta_k(2) + \cdots + \zeta_k(n)$, for $n \geq 1$ and $k = 1,2,3$

and by linearization

(9) $$\xi_k(t) \equiv ([t+1]-t)\, \xi_k([t]) + (t-[t])\, \xi_k([t+1]) ,$$

where $[x]$ represents the greatest integer less than or equal to x, and $k = 1,2,3$. Interpret $\zeta_k(n)$ as the total potential output for station k during the nth shift. Equation (9) may be interpreted as

saying that ξ_k increases at a fixed rate during each shift and that this rate varies from shift to shift. In other words, the total potential output of a shift is random but it is spread uniformly over the shift.

We now present some obvious and reasonable objections to our conventional model, with a brief rejoinder to each.

(a) It treats cumulative potential output as a continuous function of time; this is fine for continuous flow systems but not as good for discrete item manufacturing systems. Our defense in the latter case is that we will eventually restrict attention to high volume systems where individual items are more or less insignificant.

(b) The Model's representation of individual work stations through a single potential outflow process is certainly crude. In the case of stations populated by workers, for example, no formal distinction is made between single-server and multi-server stations as is common in queueing theory. These differences must somehow be expressed entirely through the distributional properties of ξ_k. Again our defense is that these fine-scale features of work center operations will be more or less insignificant for the high-volume systems of interest.

(c) With the specific distributional assumption that we have employed above, the distribution of the total potential work during shift $n+1$ is independent of all that has transpired during shifts $1,2, \ldots, n$. If the source of stochastic variability is

mechanical failure, for example, this means that the probability of a failure on day $n+1$ is unaffected by the amount of work or idleness that a machine may have experienced. This may be very unrealistic, but we will be looking at conditions under which the amount of idleness at any given station is vanishingly small, and again we argue that the structural crudeness of our model is relatively unimportant.

(d) Despite its simple appearance, our conventional model is relatively intractable, regardless of what distributional assumptions one makes about ξ. If specialized to the case of two stations, for example, it is considerably more difficult to analyze than analogous queueing models, like the M/G/1 queue with finite waiting room.

This last point does not bother us, because we do not want to analyze the conventional model. For us it is just a stepping stone to the diffusion limit. We could take limits of more finely structured, high fidelity models, but the same diffusion limit would eventually be obtained. The virtue of what we are calling our conventional model is that it makes for the simplest possible proof of the limit theorem that is our main product.

1.3 Results Obtained

In Chapter 6 we will consider a sequence of production networks indexed by $n = 1, 2, \ldots$. Here and in Chapter 6, we append a superscript n to our previous notation to indicate a quantity or process associated with the nth system. It is assumed that all buffer

sizes increase with n, and that $n^{-1/2} X^n(n\cdot) \Longrightarrow X^*$ as $n \to \infty$, where $X^{(n)}(n\cdot)$ is the net potential input process for our n^{th} system, X^* is a vector Brownian motion (with some drift vector and covariance matrix), and $\Longrightarrow$ denotes weak convergence in an appropriate function space. As our main result, it will be shown that

$$(Z^{(n)}, Y^{(n)}) \Longrightarrow (Z^*, Y^*) \qquad \text{as } n \to \infty,$$

where $Z^{(n)}$ and $Y^{(n)}$ are <u>normalized versions</u> (see equations (6.4) and (6.5)) of the contents process and lost potential output process of our n^{th} system, Z^* is a certain vector diffusion (reflected Brownian motion), and Y^* is a continuous increasing vector process associated with Z^*.

In order to say more about the diffusion limit Z^*, let us again consider the simple assembly or blending operation pictured in Figure 1. The state space for the corresponding diffusion limit Z^* is a rectangle S pictured in Figure 3. In the interior of S, Z^* behaves like the Brownian motion X^*. At the boundary, Z^* reflects instantaneously, the direction of reflection being constant along each boundary surface as pictured in Figure 3. The meaning of this boundary behavior will be explained, and the processes Z^* and Y^* will be precisely defined, in Chapter 4.

As a final task, we will begin development in Chapter 5 of the analytical theory associated with our diffusion limit Z^*. To explain the character of this theory, let us restrict discussion here to the case pictured in Figure 3, denoting by $c = (c_i)$ and $A = (a_{ij})$ the drift vector and covariance matrix respectively of the underlying

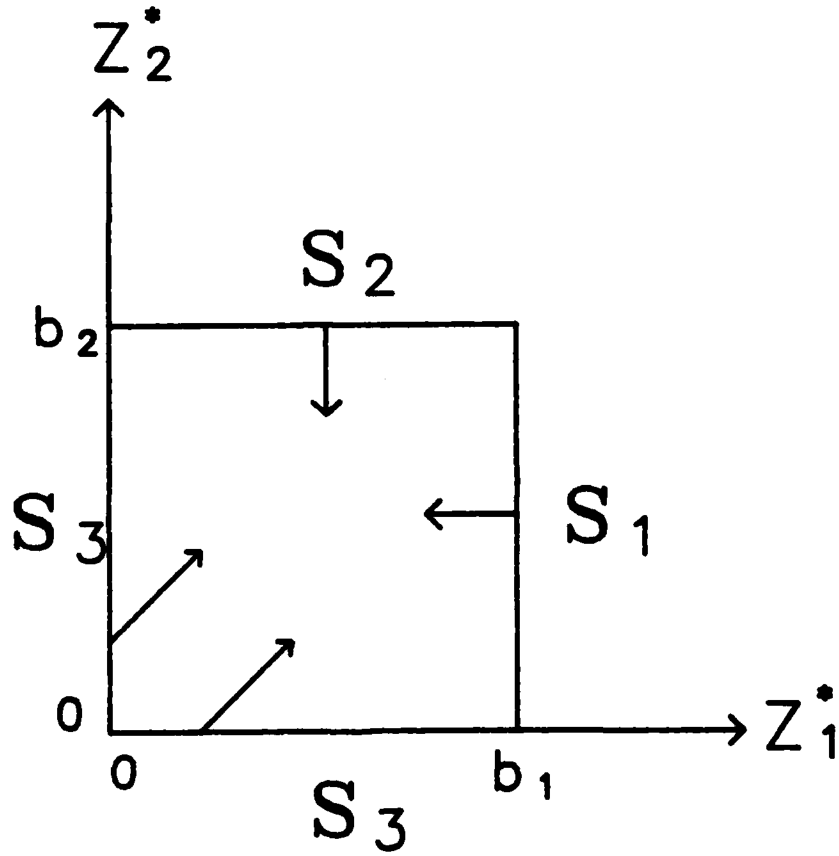

Figure 3. State Space S and Directions of Reflection for the Diffusion Limit Z^* Corresponding to the Assembly System Pictured in Figure 1.

Brownian motion X^*. Also for simplicity, we restrict current discussion to the existence and computation of the steady-state distribution for Z^*, although some other analytical problems are discussed in Chapter 5. Let us define the (constant coefficient) differential operators

$$(10) \qquad L \equiv \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{a_{ij}}{2} \frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^{2} c_i \frac{\partial}{\partial x_i} \ .$$

$$(11) \qquad D_1 = \frac{-\partial}{\partial x_1}, \qquad D_2 = \frac{-\partial}{\partial x_2} \ ,$$

and

$$(12) \qquad D_3 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \ .$$

Note that L is the elliptic operator associated with the underlying Brownian motion X^*, while D_k is a directional derivative in the direction of reflection associated with boundary surface S_k in Figure 3. In Chapter 5 it will be shown that Z^* has a unique stationary distribution Π, and that Π satisfies the <u>stationary equation</u>

$$(13) \qquad 0 = \int_S Lf(x)\ \Pi(dx) + \sum_{k=1}^{3} \int_{S_k} D_k\ f(x)\ \nu_k(dx)$$

for all $f \in C^2(S)$, where ν_1, ν_2, ν_3 are certain <u>boundary measures</u> concentrating all their mass on S_1, S_2, S_3, respectively.

It is essentially here the story ends. We conjecture that (13) uniquely determines both Π and the boundary measures ν_k. No

proof will be offered, nor do we suggest any practical scheme for computing moments of Π or other interesting quantities from (13). Nonetheless, work is currently under way on these problems, and there is reason to believe that they will be resolved in the not-too-distant future.

CHAPTER 2

PRELIMINARIES

In this chapter we present some conventions and results used throughout this work. In section three of this chapter we prove a probability result of independent interest.

2.1. Conventions and Notations

Within each chapter, we will refer to a numbered display by its number. When referring to a numbered display in another chapter, we will prefix the chapter number to the display number. For example, Theorem 4.1 refers to the first display of Chapter 4.

Due to typographical considerations we will not use special notation for vectors or vector functions. For example, 0 may mean either the real number 0 or the vector $(0, 0, \ldots, 0)$ depending on the context.

The symbol $\mathbb{R}$ will denote the real numbers. For example, $\mathbb{R}^N$ is the space of N-dimensional real vectors. The interval $[0,\infty)$ will be denoted by $\mathbb{R}_+$ and $\mathbb{R}_+^N$ will refer to the N-fold product of $\mathbb{R}_+$.

If $a \in \mathbb{R}^N$ then $\|a\| \equiv \max\{|a_1|, \ldots, |a_N|\}$. This notation will hold unless otherwise specified. Let $f = (f_1, f_2, \ldots, f_n)$ where $f_k\colon [0,T] \to \mathbb{R}$. Define

$$(1) \qquad \|f\| \equiv \sup_{0 \le t \le T} \|f(t)\| ,$$

where the value of $T \in \mathbb{R}^+$ will be clear from the context.

If X is an arbitrary space and $f: X \to R$, then $\|f\|$ is defined as

$$\|f\| \equiv \sup_{x \in X} |f(x)| \quad . \tag{2}$$

If X has a topology, we define $C(X)$ as

$$C(X) = \{f: X \to R \text{ such that } f \text{ is continuous}\} \tag{3}$$

If $a,b \in R^N$ then $a \geq b$ is defined by

$$a \geq b \text{ if and only if } a_n \geq b_n \text{ for } n = 1, 2, \ldots, N \ . \tag{4}$$

We define $a > b$ to mean $a \geq b$ and $a \neq b$. If f,g map X to R^N then $f \geq g$ means that $f(x) \geq g(x)$ for every $x \in X$. $f > g$ means that $f \geq g$ and $f \neq g$.

The symbols $\wedge$ and $\vee$ are used to represent the infimum and supremum operations respectively.

Let C^N be the space of continuous functions $f: R_+ \to R^N$. The pair $(\Omega,\underline{F})$ will always denote a measurable space and ω a generic element of Ω. Suppose $X: \Omega \to C^N$.

(5) Convention. $X(\omega)(t)$ will be denoted primarily by $X(t)$ and occasionally by $X(t,\omega)$.

Let $\tau: \Omega \to [0,\infty]$ and define $X(\tau)$ to mean

$$(6) \qquad X(\tau) = \begin{cases} X(\tau(\omega), \omega) & \text{for } \tau(\omega) < \infty \\ 0 & \text{for } \tau = \infty \end{cases} .$$

This last convention is implicitly used in Chapters 2 and 5.

2.2. Some Propositions in Real Analysis

In Propositions 7 and 11, f and g will denote continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$.

(7) Proposition. If f and g are increasing and together satisfy

$$(8) \qquad f \geq g \qquad \text{on } [s,t] ,$$

and

$$(9) \qquad f(s) = g(s) \quad \text{and} \quad f(t) - g(t) > 0 .$$

Then there exists $u \in [s,t]$ such that f is increasing at u and $f(u) > g(u)$.

Proof. If t is a point of increase for f then $u = t$ satisfies our hypothesis. Otherwise define u as

$$(10) \qquad u \equiv \operatorname{Sup}\{w \in [s,t];\ f \text{ is increasing at } w\} .$$

Line (10) implies that $f(u) = f(t)$. Furthermore, since the points of increase form a closed set, it follows that f is increasing at u. Finally, observe that $f(u)-g(u) = f(t)-g(u) \geq f(t)-g(t) > 0$. Q.E.D.

(11) Proposition. If g is continuous and f is defined by $f(t) \equiv \mathrm{Sup}_{0 \leq s \leq t}\, g(t)$, then

(12) f is a continuous function,

and

(13) $f(t) = \sup_{s \in Q_t} g(s)$, where $Q_t = \{r \in [0,t]: r$ is rational$\}$,

and

(14) If f is increasing at t then $f(t) = g(t)$.

Proof. Parts (12) and (13) follow easily from the continuity of g, and therefore only (14) will be proved. Suppose that f is increasing at t; then every $\varepsilon > 0$ satisfies $f(t+\varepsilon) > f(t-\varepsilon)$. It therefore follows that

(15) $$f(t+\varepsilon) = \sup_{s \in [t-\varepsilon, t+\varepsilon]} g(s) .$$

Line (15) and the continuity of f and g together imply that $f(t) = g(t)$. Q.E.D.

Let $(\Omega, \underline{F})$ be a measurable space and let $\{\underline{F}_t,\ t \geq 0\}$ be a filtration of $\underline{F}$. Suppose that $X, Y: \Omega \to C$ and that $X(t)$, $Y(t) \in \underline{F}_t$. Define $W(t) \equiv \mathrm{Sup}_{0 \leq s \leq t}(X(s) - Y(s))^+$.

(16) Proposition. The process $W(t)$ is adapted to $\{\underline{F}_t, t \geq 0\}$, i.e., $W(t) \in \underline{F}_t$ for $t \geq 0$.

Proof. Elementary measure theory implies that for $s \leq t$

$$(X(s) - Y(s))^+ \in \underline{F}_t \quad . \tag{17}$$

Furthermore, the space of $\underline{F}_t$-measurable functions is closed under the operation of countable supremum; hence lines (13) and (17) imply that $W(t) \in \underline{F}_t$. Q.E.D.

Let $(\underline{U}, \underline{F})$ be a measurable space. Suppose that $\{P^x, x \in X\}$ is a set of probability measures on $(\underline{U},\underline{F})$, where X is a metric space with metric d. Let $P^x(f) \equiv \int_{\underline{U}} f(u)\, dP^x(u)$.

(18) Proposition. Suppose there exists $y \in X$ such that

$$\lim_{x \to y} \sup_{Q \in \underline{F}} \left| P^x(Q) - P^y(Q) \right| = 0 \; . \tag{19}$$

Then

$$\lim_{x \to y} \sup_{f \in A} \left| P^x(f) - P^y(f) \right| = 0 \; , \tag{20}$$

where $A \equiv \{f \in \underline{F}$ such that $|f| \leq 1\}$.

Proof. Let g_x, g_y be the Radon-Nikodym derivatives of P^x, P^y relative to the measure $P^x + P^y$. Then every $f \in A$ satisfies

$$(21) \qquad P^x(f) - P^y(f) = \int_{\underline{U}} f(u)(g_x(u) - g_y(u))\,(P^x+P^y)(du)\ .$$

Line (21) implies that

$$(22) \qquad \left|P^x(f) - P^y(f)\right| \le \int_{\underline{U}} \left|g_x(u) - g_y(u)\right|\,(P^x+P^y)(du)$$

$$\le 2 \sup_{Q \in \underline{F}} \left|P^x(Q) - P^y(Q)\right|\ .$$

Lines (19) and (22) together imply (20). Q.E.D.

Let $X(t)$ be driftless Brownian motion on the real line.

(23) *Convention.* Let P^x be the distribution on the path space of X corresponding to initial state $X(0) = x$.

Let $r > 0$ be given and define τ by

$$(24) \qquad \tau \equiv \inf\{s: |X(s)| = r\}\ .$$

(25) Proposition. Let $x \in (-r,r)$ be given. Then there exists a density $f_x(s,z)$ such that

(26) $$P^x(\tau \leq t,\ X(\tau) = z) = \int_0^t f_x(s,z)\, ds \qquad \text{for} \quad t \geq 0$$

and $|z| = r$, and

(27) $f_x(s,z)$ is a continuous function of x on the interval $(-r,r)$.

Proof. Consider the following rescaling of X:

(28) $$X^*(t) \equiv X(4r^2 t)/(2r) \qquad \text{for} \quad t \geq 0 .$$

Define the stopping time T as follows:

(29) $$T \equiv \inf\{s\colon |X^*(s)| = \tfrac{1}{2}\} .$$

Lines (24), (28) and (29) together imply the identity

(30) $$(T,\ X^*(T)) = (\tau/(4r^2),\ X(\tau)/(2r)) .$$

Let $g_y(s,z)$ denote the density of $(T, X^*(T))$ corresponding to the initial state $X^*(0) = y$. Line (30) implies the following equation

(31) $$f_x(s,z) = g_{x/2r}(s/(4r^2),\ z/(2r))/(4r^2) .$$

It now suffices to show that $(T, X^*(T))$ has a density $g_x(s,z)$ which is continuous in $x \in (-1/2, 1/2)$. Observe that X^* is driftless Brownian motion with $X^*(0) = X(0)/(2r)$. Thus the symmetry of driftless Brownian motion implies the equation

$$(32) \qquad g_x(s, \tfrac{1}{2}) = g_{-x}(s, -\tfrac{1}{2}) \qquad \text{for } x \in (-\tfrac{1}{2}, \tfrac{1}{2}) .$$

Therefore we need only show that $g_x(s, -1/2)$ exists and is continuous in x. On page 267 of Ito and McKean [8] it is shown that $g_x(s, -1/2)$ exists and has the following expansion

$$(33) \qquad g_x(s, -\tfrac{1}{2}) = \sum_{n=1}^{\infty} n\,\pi \exp(-n^2 \pi^2 s/2) \sin(n\pi(x + \tfrac{1}{2})) \quad .$$

The Weierstrass M-test shows that $g_x(s, -1/2)$ is a continuous function of x. Q.E.D.

Let $r > 0$ and $K > 1$ be given. Define B_r as follows

$$(34) \qquad B_r \equiv \{x \in R^k;\ \|x\| = r\} \quad .$$

Define $R_{k,j}$ a subset of B_r by

$$(35) \quad R_{k,j} \equiv \{x \in B_r;\ x_k = jr\}, \text{ where } k = 1, \ldots, K, \text{ and } j = -1, 1 .$$

We now define a measure λ on B_r by

$$\lambda(A) = \sum_{j=-1}^{1} \sum_{k=1}^{K} \int_{R_{k,j}} I_A(x_1, x_2, \ldots, x_K)\, dx_1 \cdots dx_{k-1}\, dx_{k+1} \cdots dx_K \,. \tag{36}$$

In other words, λ is the measure which gives $(K-1)$-dimensional Lebesgue measure to each face $R_{k,j}$ of B_r.

Let $W(t)$ be standard K-dimensional Brownian motion. Define the optional time τ by

$$\tau \equiv \inf\{s\colon W(s) \in B_r\} \,. \tag{37}$$

(38) Proposition. Let $\|x\| < r$ be given. Then there exists a density $h_x(s,z)$ such that

$$P^x((\tau, W(\tau)) \in Q) = \int_Q h_x(s,z)\, d\lambda(z) \times ds \,, \tag{39}$$

where Q is a measurable subset of $[0,\infty) \times B_r$, and

$$\lim_{x \to 0} h_x(s,z) \to h_0(s,z) \qquad \text{for all } s \geq 0 \text{ and } z \in B_r. \tag{40}$$

Proof. Observe that if we define $\tau_{k,j}$ by

$$\tau_{k,j} \equiv \inf\{s\colon W_k(s) = jr\} \qquad \text{for } k = 1,2,\ldots,K \text{ and } j = -1,1, \tag{41}$$

and

$$\tau_k \equiv \bigwedge_{j=-1}^{1} \tau_{k,j} \qquad \text{for } k = 1, 2, \ldots, K, \tag{42}$$

then

(43) $$\tau = \bigwedge_{j=-1}^{1} \bigwedge_{k=1}^{K} \tau_{k,j} = \bigwedge_{k=1}^{K} \tau_k .$$

Since the $\tau_{k,j}$ are independent continuous random variables, it follows that

(44) $$P^x((\tau, W(\tau)) \in Q) = \sum_{j=-1}^{1} \sum_{k=1}^{K} P^x([\tau, W(\tau)) \in Q] \cap [\tau = \tau_{k,j}])$$

$$= \sum_{j=-1}^{1} \sum_{k=1}^{K} P^x((\tau, W(\tau)) \in Q_{k,j}) ,$$

where $Q_{k,j} \equiv Q \cap [0,\infty) \times R_{k,j}$.

It thus suffices to show that we can define a density $h_x(s,z)$ which satisfies (40) and

(45) $$P^x((\tau, W(\tau)) \in Q_{k,j}) = \int_{Q_{k,j}} h_x(s,z)\, d\lambda(z) \times ds .$$

Due to complete symmetry we can restrict our attention to $R_{1,1}$. We need the following result from Feller [4]. Define $Q_t(u,z)$ as

(46) $$Q_t(u,z) = (2\pi)^{-1/2} \sum_{n=-\infty}^{\infty} \{\exp(-\frac{(z-u+4nr)^2}{2t}) - \exp(-\frac{(z+u+(4n+2)r)^2}{2t})\} .$$

Then

(47) $$P^x(W_k(t) \in A, \tau_k > t) = \int_A Q_t(x_k,v)\, dv ,$$

where A is a measurable subset of $(-r,r)$. Lines (43) and (47) together imply that

$$(48) \quad P^x((\tau, W(\tau)) \in Q_{1,1}) = \int_{Q_{1,1}} f_{x_1}(s,z_1)\, Q_s(x_2,z_2) \cdots Q_s(x_K,z_K)\, \lambda(dz) \times ds \ .$$

Observe that $h_x(s,z)$ can be defined by

$$(49) \qquad h_x(s,z) = \sum_{k=1}^{K} I_{\{r\}}(|z_k|)\, f_{x_k}(s,z_k) \prod_{\substack{j=1\\ j\neq k}}^{K} Q_s(x_j,\, z_j) \ .$$

The density $h_x(s,z)$ then satisfies (45). Since $Q_t(x,z)$ and $f_x(s,z)$ are continuous at $x = 0$, so must be $h_x(u,z)$. Q.E.D.

Let $X(t) \equiv W(t) + ct$ and define $\tau \equiv \inf\{s\colon \|X(s)\| = r\}$.

(50) <u>Proposition</u>. Let $\|x\| < r$ be given. Then there exists a density $h^*_x(s,z)$ such that

$$(51) \qquad P^x((\tau, X(\tau)) \in Q) = \int_Q h^*_x(s,z)\, d\lambda(z) \times ds \ ,$$

and

$$(52) \qquad \lim_{x \to 0} h^*_x(s,z) \to h^*_0(s,z) \qquad \text{for} \quad s \geq 0 \quad \text{and} \quad z \in B_r \ .$$

<u>Proof</u>. Observe that τ is a finite stopping time for all x such that $\|x\| < r$. Consequently the Wald likelihood ratio argument implies

(53) $P^x((\tau, x(\tau)) \in Q) = \int_Q \exp\{-(\frac{1}{2} c'cs - c'z)\} h_x(s,z)\, d\lambda(z) \times ds$.

Therefore

$$h^*_x(s,z) = \exp\{-(\frac{1}{2} c'cs - c'z)\} h_x(s,z) .$$

Condition (52) is now obvious from the last equation. Q.E.D.

2.3 An Ergodic Theorem for Feller Processes

Let $(\Omega,\underline{F})$ be a measurable space and let $\{\underline{F}_t;\ t \geq 0\}$ be a right continuous filtration of $\underline{F}$. Suppose that S is a compact metric space and that $X = \{X(t),\ t \geq 0\}$ is a stochastic process from Ω to S which is adapted to $\{\underline{F}_t,\ t \geq 0\}$/Borel (S). Furthermore, suppose that the family $\{P^x,\ x \in S\}$ makes X into a Markov process, i.e., every $A \in$ Borel (S) satisfies

(54) $P^x(X(t+s) \in A | \underline{F}_t) = P^{X(t)}(X(s) \in A)$ a.s. P^x .

The following definition was derived from properties of Brownian motion.

(55) Definition. A state $x \in S$ is diffusion-like for X if there exists a neighborhood base $\underline{B}$ of closed sets such that every $B \in \underline{B}$ satisfies conditions (56) and (57):

(56) $P^y(\tau < \infty) = 1$ for $y \in B$,

where $\tau \equiv \inf\{s;\ X(s) \in \partial B\}$.

(57) $$\lim_{y \to x} \sup_{A \in \underline{A}} \left| P^y((\tau, X(\tau)) \in A) - P^x((\tau, X(\tau)) \in A) \right| = 0 ,$$

where $\underline{A}$ is the Borel subsets of the product topology on $R_+ \times \partial B$ induced by the usual topology on R_+ and the relativization of the topology on S to ∂B.

(58) Remark. Suppose that the following conditions are met:

$$P^y((\tau, X(\tau)) \in A) = \int_A p_y(t,z)\, \lambda(dt \times dz) \qquad \text{for } y \text{ near } x ,$$

and

$$\lim_{y \to x} p_y(t,z) = p_x(t,z) \qquad \text{a.s. } \lambda .$$

Then Scheffé's Lemma (see Scheffé [13]) implies that (57) will be satisfied.

(59) Theorem. Let X be as described above. In addition, suppose that X is a Feller process such (for a definition of Feller process see Breiman [2]) such that

(60) For every non-empty open set A there exists $t > 0$ such that $P^x(X(t) \in A) > 0$ for every $x \in S$,

(61) $$\lim_{\substack{y \to x \\ s \to t}} E^y(f(X(s))) = E^x(f(X(t))) \quad \text{whenever } f \in C(S) ,$$

and

(62) The diffusion-like points of X are dense in S.

Then X is an ergodic process.

The rest of this section is devoted to proving Theorem 59.

(63) <u>Proposition</u>. Let X be a Feller process on S such that

(64) For every measurable set A with non-empty interior A^0 there exists $u > 0$ such that $P^x(X(u) \in A) > 0$ for every $x \in S$.

Then for every such A there exists $\alpha < 1$ such that

(65) $$P^x(\tau > t) \leq \alpha^t \quad \text{for} \quad t \geq 2u \quad \text{and} \quad x \in S \; ,$$

where $\tau \equiv \inf\{s: X(s) \in A\}$.

<u>Proof</u>. Let u satisfy (64) for A^0, and let $f(x) = P^x(X(u) \in A^0)$. Since X is a Feller process and A^0 open it follows that

(66) $$\underline{\lim}_{y \to x} f(y) = \underline{\lim}_{y \to x} P^y(X(u) \in A^0) \geq P^x(X(u) \in A^0) = f(x) \quad .$$

Therefore f is lower semi-continuous on S. Since S is compact, f achieves its minimum at $x_0 \in S$. Thus $\inf_{x \in S} f(x) = f(x_0) \equiv 1-\rho > 0$.

Let $t \geq 2u$ be given. Set $n \equiv [t/(2u)]/2$. Using an obvious induction observe that

(67) $$P^x(\tau > t) \leq P^x(X(u) \in A^c, \ldots, X(2nu) \in A^c)$$
$$= E^x[I_{A^c}(X(u))P^{X(u)}(X(u) \in A^c, \ldots, X((2n-1)u) \in A^c)]$$
$$\leq E^x[I_{A^c}(X(u))\ \rho^{2n-1}] \leq \rho^{2n} \leq \alpha^t ,$$

where $\alpha = \rho^{1/(2u)}$. Q.E.D.

(68) Proposition. Let X be a Feller process on S which satisfies (61) and let D be a closed subset of S. Suppose x is a diffusion-like point and that $x \in D^c$. Then the function $P_t(y,D) \equiv P^y(X(t) \in D)$ is continuous at x uniformly in t, i.e., for $\varepsilon > 0$ there exists a closed neighborhood F of x such that

(69) $$\sup_{t \geq 0} \sup_{x_1,x_2 \in F} |P_t(x_1,D) - P_t(x_2,D)| \leq \varepsilon .$$

Proof. Since x is diffusion-like, x has a closed neighborhood B disjoint from D that satisfies (56) and (57). Define $\tau \equiv \inf\{s: X(s) \in \partial B\}$. Since X has continuous paths and ∂B is closed, it follows that τ is a stopping time. Furthermore, pathwise continuity implies

(70) $$P_t(y,D) = P^y(X(t) \in D, \tau < t) \qquad \text{for} \quad y \in B .$$

Apply the strong Markov property to (70) to obtain

(71) $$P_t(y,D) = E^y(P_{t-\tau}(X(\tau), D)) .$$

Condition (61) and the fact that D is closed together imply that $P_t(y,D)$ is upper semi-continuous in (t,y) and thus measurable. Furthermore $\mathrm{Sup}_{t,y} \left|P_t(y,D)\right| \leq 1$. Therefore Proposition (18) can be applied to (71) to obtain (69).

(72) <u>Proposition</u>. Let X satisfy the conditions of Theorem (59). Then every closed subset D satisfies

(73) $$\lim_{t \to \infty} \sup_{x_1, y_2 \in S} \left|P_t(x_1,D) - P_t(x_2,D)\right| = 0 \quad .$$

<u>Proof</u>. For $D = S$ the proposition is trivial, so suppose D^c is a non-empty subset of S. Condition (62) implies that there exists $x \in D^c$ which is diffusion-like. Let $\varepsilon > 0$ be given. Proposition (68) implies that there exists F, a closed neighborhood of x, which satisfies (69). Let $x_1, x_2 \in S$, and let X^1, X^2 be independent versions of X such that $X^1(0) = x_1$ and $X^2(0) = x_2$. Define T as follows:

(74) $$T \equiv \inf\{s\colon (X^1(s), X^2(s)) \in F \times F\} \quad .$$

Observe that

(75) $$P_t(x_1,D) = P(X^1(t) \in D,\ T > t) + P(X^1(t) \in D,\ T \leq t) \quad ,$$

and

(76) $$P_t(x_2,D) = P(X^2(t) \in D,\ T > t) + P(X^2(t) \in D,\ T \leq t) \quad .$$

It is easy to verify that the coupled process (X^1, X^2) satisfies the conditions of Proposition 63, and therefore there exists an $\alpha < 1$ such that

$$(77) \qquad \sup_{x_1, x_2 \in S} P(T > t) \leq \alpha^t \ , \quad \text{for} \quad t \geq 2u \ .$$

Lines (75), (76), and (77) together imply

$$(78) \qquad \left| P_t(x_1, D) - P_t(x_2, D) \right|$$
$$\leq \left| P(X^1(t) \in D,\ T \leq t) - P(X^2(t) \in D,\ T \leq t) \right| + \alpha^t \ .$$

Since T is a Markov time it follows that

$$(79) \qquad P(X^1(t) \in D,\ T \leq t) = E[I_{\{T \leq t\}}\, P(X^1(t) \in D \,|\, \underline{F}_T)]$$

$$\text{(where } \underline{F}_T \equiv \sigma((X^1(s), X^2(s)),\ 0 \leq s \leq T))$$

$$= E[I_{\{T \leq t\}}\, P_{t-T}(X^1(T),\ D)] \ .$$

It similarly follows that

$$(80) \qquad P(X^2(t) \in D,\ T \leq t) = E[I_{\{T \leq t\}}\, P_{t-T}(X^2(T),\ D)] \ .$$

Line (69) now implies that

$$(81)\qquad \left|P(X^2(t)\in D,\ T\le t)-P(X^1(t)\in D,\ T\le t)\right|$$
$$\le E[I_{\{T\le t\}}\left|P_{t-T}(X^2(T),\ D)-P_{t-T}(X^1(T),\ D)\right|]$$
$$\le E[I_{\{T\le t\}}\ \varepsilon]\le\varepsilon\ .$$

Therefore it follows from (78) and (81) that

$$(82)\qquad \sup_{x_1,x_2\in S}\left|P_t(x_1,D)-P_t(x_2,D)\right|\le\varepsilon+\alpha^t\ .$$

Since ε is arbitrary, equation (73) follows easily from (82).

(83) <u>Proposition</u>. Let $P_t(x,\cdot)$ be a Markov kernel on a compact metric space S. Suppose $P_t(x,\cdot)$ satisfies (73). Then there exist $\{t_n,\ n\ge 1\}$ such that $t_n\uparrow\infty$ and a probability measure Π on S which together satisfy

$$(84)\qquad \lim_{n\to\infty}\ \sup_{x\in S}\left|P_{t_n}(x,f)-\Pi(f)\right|=0\qquad\text{for } f\in C(S)\ .$$

<u>Proof</u>. Fix $y\in S$. Since S is compact, the family of measures $\{P_t(y,\cdot),\ t\ge 0\}$ is tight. Therefore, there exist $\{t_n,\ n\ge 1\}$ and a probability measure Π which together satisfy

$$(85)\qquad P_{t_n}(y,\cdot)\Longrightarrow\Pi(\cdot)\qquad\text{as } n\to\infty,$$

and

$$(86)\qquad t_n\uparrow\infty\qquad\text{as } n\to\infty.$$

Let $f \in C(S)$ such that $0 < f \leq 1$. Line (85) implies that

$$(87) \qquad P_{t_n}(y,f) \to \Pi(f) \qquad \text{as} \quad n \to \infty .$$

Let m be an arbitrary integer. Set $D_k \equiv \{x;\ k-1 < mf(x) \leq k\}$, and

$$(88) \qquad f_m \equiv \sum_{k=1}^{m} k/m \ I_{D_k} .$$

Lines (73), (85) and (86) together imply that there exists n_0 such that

$$(89) \qquad \sup_{x \in S} \bigvee_{k=1}^{m} \left| P_{t_n}(x,\ D_k) - P_{t_n}(y,\ D_k) \right| \leq 1/m^2 \qquad \text{for} \quad n \geq n_0 ,$$

and

$$(90) \qquad \left| P_{t_n}(y,f) - \Pi(f) \right| \leq 1/m \qquad \text{for} \quad n \geq n_0 .$$

Therefore, the triangle inequality implies

$$(91) \quad \begin{aligned} \left| P_{t_n}(x,f) - \Pi(f) \right| &\leq \left| P_{t_n}(x,f - f_m) \right| + \left| P_{t_n}(x,f_m) - P_{t_n}(y,f_m) \right| \\ &\quad + \left| P_{t_n}(y,f_m - f) \right| + \left| P_{t_n}(y,f) - \Pi(f) \right| \\ &\leq 1/m + m/m^2 + 1/m + 1/m = 4/m \qquad \text{for} \quad n \geq n_0. \end{aligned}$$

The arbitrariness of m implies (84) for $f \in C(S)$ such that $0 < f \leq 1$. The general case now follows easily from the compactness of S.

(92) <u>Proposition</u>. Let X be a Markov process on a compact metric space S. Suppose X satisfies (60), (61), and (84). Then X is an ergodic process.

<u>Proof</u>. We need to show that

(93) $$\lim_{t \to \infty} P_t(x,f) = \Pi(f) \quad \text{for } x \in S \text{ and } f \in C(S) ,$$

and

(94) $$\Pi(f) > 0 \quad \text{for } f \in C(S) \text{ and } f > 0 .$$

Let $f \in C(S)$ and let $\varepsilon > 0$ be given. Condition (84) implies that there exists s such that

(95) $$\sup_{x \in S} \left|P_s(x,f) - \Pi(f)\right| \leq \varepsilon .$$

Let $t > s$ be given. Then every $y \in S$ satisfies

(96) $$\begin{aligned}\left|P_t(y,f) - \Pi(f)\right| &\leq \int_S P_{t-s}(y, dx) \left|P_s(x,f) - \Pi(f)\right| \\ &\leq \int_S P_{t-s}(y, dx)\varepsilon = \varepsilon .\end{aligned}$$

Therefore equation (93) must hold.

Finally, suppose $f \in C(S)$ and $f > 0$. Observe that condition (61) implies that $P_t(\cdot,f) \in C(S)$. Therefore (93) implies

(97) $$\begin{aligned}\pi(f) &= \lim_{s \to \infty} P_{s+t}(x,f) \\ &= \lim_{s \to \infty} \int_S P_s(x,dy)\, P_t(y,f) = \int_S \Pi(dy)\, P_t(y,f) .\end{aligned}$$

Lines (60) and (97) together imply (94). Q.E.D.

Theorem 59 can thus be proved by applying in sequence Propositons 72, 83 and 92.

2.4 Weak Convergence in Function Space

Let $(\Omega, \underline{B}, P)$ be a probability space. Let (M,d) be a metric space and let $\underline{F}$ be the Borel sets of M.

(98) Definition. Let X be a mapping from Ω to M such that $X^{-1}(\underline{F}) \subset \underline{B}$. Then we say that X is an M-valued random variable on Ω.

(99) Remark. We denote the above relationship by either $X \in \underline{B}/\underline{F}$ or $X \in \underline{B}$. We use the latter notation only if $\underline{F}$ is clear from the context.

(100) Definition. Let $\{X_n, n \geq 1\}$ be a sequence of M-valued random variables defined on Ω. Suppose that every closed subset D of M satisfies

(101) $$\overline{\lim_{n \to \infty}} P(X_n \in D) \leq P(X_\infty \in D) \quad ,$$

for some M-valued random variable X_∞. We then say that $\{X_n, n \geq 1\}$ converges weakly to X_∞. We denote this by

(102) $$X_n \Rightarrow X_\infty \text{ as } n \to \infty$$

Consider the particular case where $M = C^N$, d is the metric given in Whitt [14], and $\underline{F}$ is the σ-field of Borel sets on C^N.

For $T > 0$ define $r_T \colon C^N \to C^N[0,T]$ as follows

(103) $$r_T(x)(s) = x(s) \qquad \text{for } x \in C^N \text{ and } s \in [0,T] .$$

(104) Lemma. Let $\{X_n, n \geq 1\}$ be C^N-valued random variables. Then $\{X_n, n \geq 1\}$ converges weakly to X_∞ if and only if for every $T > 0$ the sequence $\{r_T(X_n), n \geq 1\}$ converges weakly to $r_T(X_\infty)$.

For a proof of Lemma 104, see Whitt [14].

CHAPTER 3

PRODUCTION NETWORKS

In this chapter we define precisely our general production network models, examples of which were discussed in Chapter 1. In the first section we describe the deterministic features of such systems. In Sections 2 and 3 we propose two different ways of modelling the stochastic structure of a production network.

3.1 The General Model

We begin our description of the general production system by specifying its network structure. The network consists of $K+1$ stations indexed by the set $k = 1, 2, \ldots, K+1$. Stations $1, 2, \ldots, L$ $(L \leq K)$ are called external stations. We specify the system's flow structure by a map $\sigma: \{1, 2, \ldots, K\} \to \{L+1, \ldots, K+1\}$. Interpret $\sigma(k)$ as the successor station at which output from station k is used as input. Thus

$$(1) \qquad \Pi(k) \equiv (j \in \{1, \ldots, K\}; \sigma(j) = k)$$

is the set of predecessors whose output is used directly as input at station k $(k = L+1, \ldots, K+1)$.

It is assumed that $\sigma(k) > k$, and σ maps $\{1, 2, \ldots, K\}$ onto $\{L+1, \ldots, K+1\}$. These two conditions guarantee that our network will be an arborescent structure whose terminal station is $K+1$. Thus inputs from the external stations $\{1, 2, \ldots, L\}$ are combined by stages into inputs for the terminal station $K+1$.

It is assumed that for each station k $(k < K+1)$ there exists a finite output buffer of size b_k. Furthermore, we associate a potential output process $\xi_k = (\xi_k(t),\ t \geq 0)$ with each station k. The process ξ_k is assumed to be continuous and increasing with $\xi_k(0) = 0$. Interpret $\xi_k(t)$ as the total production of station k through time t, providing that station k works without impediment during the time interval $[0,t]$. We now define the <u>net potential input</u> process

$$(2) \qquad X_k(t) \equiv \xi_k(t) - \xi_{\sigma(k)}(t) \qquad \text{for} \quad t \geq 0 \quad \text{and} \quad k = 1, 2, \ldots, K.$$

Interpret $X_k(t)$ as the potential change in the kth buffer's inventory from time 0 to time t. Thus, if we let $Z_k(t)$ denote the kth buffer's <u>actual</u> inventory at time t, then we have

$$(3) \qquad Z_k(t) = Z_k(0) + X_k(t) \ , \qquad t \geq 0 \ ,$$

providing that neither station k nor station $\sigma(k)$ is impeded during the interal $[0,t]$. We need to modify equation (3) so that it is unconditionally valid. Thus we now introduce the <u>lost potential output</u> process $Y_k(t)$. The process $Y_k(t)$ should be construed as the amount of potential output station k loses due to either starvation or blockage during $[0,t]$. We can now express the <u>actual</u> output process for station k by the difference $\xi_k(t) - Y_k(t)$. Therefore, the <u>actual</u> input process to buffer k is given by the expression $\xi_k(t) - Y_k(t) - (\xi_{\sigma(k)}(t) - Y_{\sigma(k)}(t))$. This simplifies to

$X_k(t) - Y_k(t) + Y_{\sigma(k)}(t)$. Thus the proper generalization of equation (3) is

$$(4) \quad Z_k(t) \equiv Z_k(0) + X_k(t) - Y_k(t) + Y_{\sigma(k)}(t) \ , \quad t \geq 0 \text{ and } k \leq K.$$

We denote by Z and Y the vector processes $(Z_1, Z_2, \ldots, Z_K)$ and $(Y_1, Y_2, \ldots, Y_{K+1})$ respectively. Of course the issue of how to define Y precisely has again been skirted. We begin with the reasonable requirement that

$$(5) \quad Y_k \text{ is continuous and increasing, with } Y_k(0) = 0 \text{ for all } k .$$

Furthermore, the intended meanings of Y and Z suggest that they jointly satisfy

$$(6) \quad 0 \leq Z_k(t) \leq b_k \qquad \text{for } k = 1, 2, \ldots, K \text{ and all } t \geq 0.$$

$$(7) \quad \int_0^t [b_k - Z_k(s)] \, dY_k(s) = 0 \quad \text{for } k = 1, \ldots, L \text{ and all } t \geq 0.$$

$$(8) \quad \int_0^t \Big[(b_k - Z_k(s)) \wedge \bigwedge_{j \in \Pi(k)} Z_j(s)\Big] \, dY_k(s) = 0 \quad \text{for } k = L+1, \ldots, K \text{ and all } t \geq 0,$$

$$(9) \quad \int_0^t \Big[\bigwedge_{j \in \Pi(K+1)} Z_j(s)\Big] \, dY_{K+1}(s) = 0 \qquad \text{for all } t \geq 0.$$

What we will now see is that (5)-(9) can actually be used to define Y and Z in terms of X and $Z(0)$ in precise mathematical terms, thus

completing the specification of the model's (non-probabilistic) structure. The following representation theorem will be proved in Chapter 4. Here and later we denote by S the <u>state space</u> of the contents process Z:

(10) $$S \equiv [0,b_1] \times \cdots \times [0,b_K] \equiv [0,b] \ .$$

(11) <u>Theorem</u>. Given X continuous with $X(0) = 0$ and $Z(0) \in S$, there exists a unique Y satisfying (5)-(9) with Z defined in terms of Y by (4).

To complete our model specification, we need to impose distributional assumptions on the vector process ξ. Possible distributions for ξ are presented in detail in the next two sections of this chapter.

3.2 A Random Walk Model

One way to generate the process ξ is to construct it from a sequence of IID random vectors $\{\zeta(n), n \geq 1\}$ by summation and linear interpolation. That is, define ξ_k at integer times m by the equation

(12) $$\xi_k(m) \equiv \sum_{n=1}^{m} \zeta_k(n) \quad ,$$

and define ξ_k at non-integer times t by linear interpolation, namely

$$(13) \qquad \zeta_k(t) = ([t+1]-t)\, \zeta_k([t]) + (t-[t])\, \zeta_k([t+1]) \quad .$$

Here $[x]$ denotes the greatest integer less than or equal to x.

As was mentioned in Section 1.2, one can interpret $\zeta_k(n)$ as the total potential output of station k during the nth shift. Furthermore (13) may be viewed as saying that work proceeds at the fixed rate of $\zeta_k(n)$ units per shift. Finally, note that the components of the random vector ζ need not be an independent set. However, we will assume in Chapter 6 that $\{\zeta(n), n \geq 1\}$ generates an X process which is fully K-dimensional.

It is of interest to note that if $K = 1$, then this model represents an extension of the classical discrete-time dam model to continuous time. A good reference for the dam problem is Moran [10].

3.3 A Random Environment Model

It is useful to conceive of the random rate vector $\zeta(n)$ as being determined by the "working environment" of the nth shift. This working environment concept allows us to generalize the model of Section 2 in the following way. Suppose that each working environment persists for a random period of time and that work proceeds at a fixed rate for the duration of each working environment. Furthermore we will allow for working environments to influence one another. There are many interesting ways to implement this model. For one specific alternative, consider the following model. Let $\theta = \{\theta(t), t \geq 0\}$ be a continuous-time stationary Markov chain with states $1, 2, \ldots, M$.

Let $r(1), r(2), \ldots, r(M)$ be positive $K+1$ dimensional vectors, and define

$$\text{(14)} \qquad \xi_k(t) \equiv \int_0^t r_k(\theta(s))ds \,, \quad \text{for } k = 1, \ldots, K+1 \text{ and } t \geq 0.$$

Equation (14) may be interpreted as saying that $\theta(s)$ is the state of working environment at time s, and that the kth station works at rate $r_k(\theta(s))$ at time s. Since θ is a stationary, finite state Markov chain, it is possible to recursively define the transition times to the successively visited states. That is, define $T_0 \equiv 0$ and T_n recursively by

$$\text{(15)} \qquad T_n \equiv \inf\{s > T_{n-1};\ \theta(s) \neq \theta(T_{n-1})\} \,, \qquad \text{for } n \geq 1 \;.$$

Now define the nth holding time τ_n by

$$\text{(16)} \qquad \tau_n \equiv T_{n+1} - T_n \,, \qquad \text{for } n \geq 0 \;.$$

Furthermore, there exist strictly positive constants $\lambda(m)$, $1 \leq m \leq M$, and an $M \times M$ transition matrix Q which jointly satisfy

$$\text{(17)} \qquad P(\theta(T_{n+1}) = j,\ \tau_n > t \mid \theta(T_n) = i) = Q_{ij}\, e^{-\lambda(i)t}$$
$$\text{for } t \geq 0 \text{ and } i,j \in \{1, 2, \ldots, M\},$$

and

(18) $\quad Q_{ii} = 0 \,, \qquad$ for $i = 1, 2, \ldots, M$.

The interested reader can refer to Chapter 8 of Çinlar [3] for a proof of (17) and (18).

CHAPTER 4

THE REFLECTION MAPPING

We take as given integers $K \geq L \geq 1$, and a map σ: $\{1,2,\ldots,K\} \rightarrow \{L+1, \ldots, K+1\}$ such that $\sigma(k) > k$ and σ maps $\{1, 2, \ldots, K\}$ onto $\{L+1, \ldots, K+1\}$. Also taken as primitive is a vector $b = (b_1, \ldots, b_K)$ with $b_k > 0$ for all k. Let C^K be the space of continuous functions x: $[0,\infty) \rightarrow R^K$, endowed with the topology of uniform convergence on compact intervals. Component functions are denoted $x_j(t)$ for $t \geq 0$ and $j = 1, 2, \ldots, K$. Let C_S be the set of $x \in C^K$ such that $x(0) \in S$.

(1) Theorem. For each $x \in C_S$ there exists a unique pair of functions $y \in C^{K+1}$ and $z \in C^K$ satisfying

(2) $z_k(t) = x_k(t) - y_k(t) + y_{\sigma(k)}(t)$, $k = 1, \ldots K$ and $t \geq 0$,

(3) $y_k(\cdot)$ is increasing, with $y_k(0) = 0$ $(k = 1, \ldots, K+1)$,

(4) $z(t) \in S$, $t \geq 0$,

(5) $\int_0^t (b_k - z_k(s))\, dy_k(s) = 0$, $k = 1, \ldots, L$ and $t \geq 0$,

$$(6) \quad \int_0^t \Big[(b_k - z_k(s)) \bigwedge_{j\in\Pi(k)} z_j(s)\Big]\, dy_k(s) = 0, \quad k = L+1, \ldots, K \text{ and } t \geq 0 ,$$

and

$$(7) \quad \int_0^t \Big[\bigwedge_{j\in\Pi(K+1)} z_j(s)\Big]\, dy_{K+1}(s) = 0 \quad , \quad t \geq 0 \quad .$$

Moreover, setting $y = \psi(x)$ and $z = \phi(x)$, we have the following:

(8) Both ψ and ϕ are continuous mappings on C_S .

(9) Fix $x \in C_S$ and $T > 0$. Define $x^*(t) \equiv z(T) + x(T+t) - x(T)$, $y^*(t) \equiv y(T+t) - y(T)$, and $z^*(t) \equiv z(T+t)$. Then $y^* = \psi(x^*)$ and $z^* = \phi(x^*)$.

(10) Suppose $x = x'$ on $[0,t]$, then $\psi(x) = \psi(x')$ and $\phi(x) = \phi(x')$ on $[0,t]$.

The proof of Theorem 1 is given in Sections 1, 2 and 3. In Section 4 we produce a convenient bound for y. This bound will be of interest only in Section 5.4.

Convention. For the remainder of this chapter the symbol $\sigma(k)$, when used as a subscript, will be shortened to simply σ. For example, $y_\sigma \equiv y_{\sigma(k)}$ or $x_{\sigma(k)} \equiv x_\sigma$, and so on.

4.1 Existence and Uniqueness

We begin by proving that conditions (2)-(7) are equivalent to the conditions

$$(11)\quad y_k(t) = \sup_{0 \le s \le t} (x_k(s) + y_\sigma(s) - b_k)^+ , \quad k = 1, 2, \ldots ,L \text{ and } t \ge 0 ,$$

$$(12)\quad y_k(t) = \sup_{0 \le s \le t} \bigvee_{j \in \Pi(k)} (y_j(s) - x_j(s))^+ \vee (x_k(s)+y_\sigma(s)-b_k)^+, \quad L+1 \le k \le K \text{ and } t \ge 0 ,$$

and

$$(13)\qquad y_{K+1}(t) = \sup_{0 \le s \le t} \bigvee_{j \in \Pi(K+1)} (y_j(s) - x_j(s))^+ .$$

Suppose that y satisfies (2)-(7). To verify the forward implication we start by showing that conditions (3) and (4) imply the weaker conditions,

$$(14)\quad y_k(t) \ge \sup_{0 \le s \le t} (x_k(s) + y_\sigma(s) - b_k)^+ , \quad k = 1, 2, \ldots, K \text{ and } t \ge 0 ,$$

and

$$(15)\quad y_k(t) \ge \sup_{0 \le s \le t} \bigvee_{j \in \Pi(k)} (y_j(s)-x_j(s))^+ , \quad L \le k \le K+1 \text{ and } t \ge 0 .$$

Observe that $0 \le z_j(t)$, $j \in \Pi(k)$, implies that $y_k(t) \ge y_j(t)-x_j(t)$. Furthermore, since $y_k(t) \ge y_k(s) \ge y_k(0) = 0$ $(t \ge s \ge 0)$, it is now obvious that $y_k(t) \ge \wedge_{j \in \Pi(k)} (y_j(s)-x_j(s))^+$ for $s \in [0,t]$. The last line is equivalent to condition (15). In similar fashion, it can be shown that (3) and (4) imply (14). Note that (14) and (15) together imply

(16) $\quad y_k(t) \geq \sup_{0 \leq s \leq t} \bigvee_{j \in \Pi(k)} (y_j(s)-x_j(s))^+ \vee (x_k(s)+y_\sigma(s)-b_k)^+$

$L+1 \leq k \leq K$ and $t \geq 0$.

Fix $k \in \{L+1, L+2, \ldots, K\}$; we will now prove that (6) and (16) together imply that y_k satisfies (12). Begin by defining $\tau \equiv \sup\{t \geq 0\colon\ y_k$ satisfies (12) on $[0,t]\}$. The definition of τ implies that (12) is satisfied on $[0,\tau)$. Suppose in contradiction that $\tau < \infty$. Then by the continuity of z and y it follows that (12) is satisfied on $[0,\tau]$. By virtue of (16) and Proposition 2.7 there exists $\tau_0 > \tau$ such that τ_0 is a point of increase for y_k and

$$y_k(\tau_0) > \sup_{0 \leq s \leq \tau_0} \bigvee_{j \in \Pi(k)} (y_j(s)-x_j(s))^+ \vee (x_k(s) + y_\sigma(s) - b_k)^+.$$

The last inequality implies that $(b_k-z_k(\tau_0)) \wedge_{j \in \Pi(k)} z_j(\tau_0) > 0$, and thus the continuity of z implies that there exists $\delta > 0$ such that $(b_k-z_k(s)) \wedge_{j \in \Pi(k)} z_j(s) > 0$ for $s \in [\tau_0-\delta, \tau_0+\delta]$. Consequently we now obtain the inequality

$$\int_0^{\tau_0+\delta} (b_k-z_j(s)) \bigwedge_{j \in \Pi(k)} z_j(s)\, dy_k(s) > 0 .$$

The last inequality contradicts (6), and therefore $\tau = \infty$. Thus y_k, for $k = L+1, L+2, \ldots, K$, satisfies (12) on $[0,\infty)$. In similar fashion it can be shown that (5), (7), (14) and (15) together imply (11) and (13).

To show the reverse implication, begin by assuming that y satisfies (11)-(13). It is easy to show that (11)-(13) together imply (3) and (4). Fix $k \in \{L+1, L+2, \ldots, K\}$. The function y_k satisfies (6) if and only if the set $\{t:(b_k-z_k(t)) \wedge_{j\in\Pi(k)} z_j(t) > 0\}$ has y_k measure 0. By the continuity of z, it therefore suffices to show that y_k increases at t only if $(b_k-z_k(t)) \wedge_{j\in\Pi(k)} z_j(t) = 0$. Suppose that y_k is increasing at t, then $y_k(t+\varepsilon) > y_k(t-\varepsilon)$ for $\varepsilon > 0$. It now follows from (12) that

$$(17) \quad y_k(t+\varepsilon) = \sup_{t-\varepsilon \le s \le t+\varepsilon} \bigvee_{j\in\Pi(k)} (y_j(s)-x_j(s))^+ \vee (x_k(s)+y_\sigma(s)-b_k)^+ > y_k(t-\varepsilon) \ge 0 \quad .$$

Since $y_k(t+\varepsilon) > 0$, we can omit the "positive part operator" from (17) to obtain

$$(18) \quad y_k(t+\varepsilon) = \sup_{t-\varepsilon \le s \le t+\varepsilon} \bigvee_{j\in\Pi(k)} (y_j(s)-x_j(s)) \vee (x_k(s)+y_\sigma(s)-b_k).$$

Since x and y are continuous, we can let $\varepsilon \to 0$ in (17) to get $y_k(t) = \vee_{j\in\Pi(k)} (y_j(t)-x_j(t)) \vee (x_k(t)+y_\sigma(t)-b_k)$. Therefore $(b_k-z_k(t)) \wedge_{j\in\Pi(k)} z_j(t) = 0$. This shows that (11)-(13) imply (6). In similar fashion conditions (5) and (7) can be verified.

From now on we will only consider the question of existence and uniqueness in terms of conditions (11)-(13). Before we can begin, we need to introduce a great deal of new notation.

For $j = 1, 2, \ldots, K+1$ define a chain to station j to be an ordered set of indices $c = (\ell_1, \ell_2, \ldots, \ell_m)$ which satisfies

$$(19) \qquad 1 \leq \ell_1 \leq L , \qquad \ell_m = j ,$$

and

$$(20) \qquad \ell_{k-1} \in \Pi(\ell_k) \qquad \text{for } k = 2, \ldots, m .$$

Define $C(j)$ to be the set of all chains to j, i.e.,

$$(21) \qquad C(j) \equiv \{c: \ c \text{ satisfies (19) and (20)}\} .$$

Because $\sigma(k) > k$ the elements of any chain are necessarily distinct.

We will say that our network is M-stages long if the longest chain to $K+1$ has M elements, i.e.,

(22) $$M \equiv \max\{\mathrm{card}(c): \; c \in C(K+1)\} \; .$$

Define the stages of a network in the following way.

(23) $$\underline{S}(M) \equiv \{K+1\} \; ,$$

and

(24) $$\underline{S}(M-n) \equiv \{j: \; j \in \Pi(k) \;\text{ for some }\; k \in \underline{S}(M-n+1)\} \quad \text{for } \; n = 1, \ldots, M-1 \; .$$

Define $\underline{E}(n)$ for $n = 1, \ldots, M-1$ as follows:

(25) $$\underline{E}(n) \equiv \underline{S}(n+1) - \{1, 2, \ldots, L\} \; .$$

In words, $\underline{E}(n)$ is the set of stations which receive inputs from $\underline{S}(n)$.

We now introduce the important inequality constraints

(26) $$y_k(t) \le \sup_{0 \le s \le t} (x_k(s)+y_\sigma(s)-b_k)^+ \;\text{ for }\; k = 1,\ldots,L \;\text{ and }\; t \ge 0,$$

(27) $$y_k(t) \le \sup_{0 \le s \le t} \; \bigvee_{j \in \Pi(k)} (y_j(s)-x_j(s))^+ \vee (x_k(s)+y_\sigma(s)-b_k)^+ \quad \text{for } \; k = L+1, \ldots, K \;\text{ and }\; t \ge 0 \; ,$$

and

(28) $y_{K+1}(t) \le \sup_{0 \le s \le t} \bigvee_{j\in\Pi(k)} (y_j(s)-x_j(s))^+ \qquad \text{for} \quad t \ge 0 .$

Let $\underline{U}$ be the class of positive, continuous, increasing (component-wise) vector functions which satisfy (26)-(28).

For $\ell = 1, \ldots, M-1$ define $h^{(\ell)}: \underline{U} \to \underline{U}$ and $g^{(\ell)}: \underline{U} \to \underline{U}$ as follows:

(29) $h_k^{(\ell)}(y)(t)$

$$\equiv \begin{cases} \sup\limits_{0 \le s \le t} \bigvee\limits_{j\in\Pi(k)} (y_j(s)-x_j(s))^+ \vee y_k(s) & \text{for} \quad k \in \underline{E}(\ell) \\ y_k(t) \ , & \text{for} \quad k \notin \underline{E}(\ell) \end{cases}$$

and

(30) $g_k^{(\ell)}(y)(t)$

$$\equiv \begin{cases} \sup\limits_{0 \le s \le t} (x_k(s)+y_\sigma(s)-b_k)^+ \vee y_k(s) & \text{for} \quad k \in \underline{S}(M-\ell) \\ y_k(t) & \text{for} \quad k \notin \underline{S}(M-\ell) \end{cases}$$

(31) Remark. It is easy to verify that $h^{(\ell)}$ and $g^{(\ell)}$ preserve the inequalities (26)-(28). Furthermore, if y satisfies (11)-(13) on $[0,T]$ then $g^{(\ell)}(y) = h^{(\ell)}(y) = y$ on $[0,T]$.

Now define $H: \underline{U} \to \underline{U}$ and $G: \underline{U} \to \underline{U}$ by the relations:

$$H(y) \equiv h^{(M-1)} \circ h^{(M-2)} \circ \cdots \circ h^{(1)}(y) , \tag{32}$$

and

$$G(y) \equiv g^{(M-1)} \circ g^{(M-2)} \circ \cdots \circ g^{(1)}(y) . \tag{33}$$

(34) Remark. H and G inherit from $h^{(\ell)}$ and $g^{(\ell)}$ the properties given in Remark (31).

Our immediate goal is to show that if $y \in \underline{U}$ satisfies (11)-(13) on $[0,T]$ then $(G \circ H)(y)$ satisfies (11)-(13) on $[0, T+\delta]$ where $\delta > 0$. We begin this task with the following remark.

Remark. Suppose that y satisfies

$$\sup_{0 \leq s \leq t} \bigvee_{j \in \Pi(k)} (y_j(s)-x_j(s))^+ \leq y_k(s) \qquad \text{for } k \in \bigcup_{n=1}^{\ell-1} \underline{E}(n) \tag{35}$$

and $t \geq 0$. Then for $k \in \cup_{n=1}^{\ell} \underline{E}(n)$, $h_k^{(\ell)}(y)$ satisfies (35) for $t \geq 0$.

(36) Proposition. For $k \in \{L+1, \ldots, K+1\}$, $H_k(y)$ satisfies (35) for $t \geq 0$.

Proof. Use the previous remark recursively to deduce that $H_k(y)$ satisfies (35) for $k \in \cup_{n=1}^{M-1} \underline{E}(n)$. Finally, note that the surjectiveness of σ implies that $\cup_{n=1}^{M-1} \underline{E}(n) = \{L+1, \ldots, K+1\}$.

Define $\underline{L}_T$ to be the subset of $\underline{U}$ which satisfies (35) on $[0,T]$, i.e.,

(37) $$\underline{L}_T \equiv \{y \in \underline{U}: \text{ for } k = L+1, \ldots, K+1, \ y_k \text{ satisfies (35) on } [0,T]\} .$$

(38) <u>Proposition</u>. Suppose $y \in \underline{U}$ and

(39) $$y \text{ satisfies (11)-(13) on } [0,T] ,$$

(40) $$y \in \underline{L}_{T+\delta} ,$$

and

(41) $$\sup_{T \le s_1 \le s_2 \le T+\delta} \|x(s_1)-x(s_2)\| \le \bigwedge_{n=1}^{K} b_n .$$

Then for every $\ell \in \{1, 2, \ldots, M-1\}$, $g^{(\ell)}(y) \in \underline{L}_{T+\delta}$.

<u>Proof</u>. We need to prove that for $k = L+1, \ldots, K+1$, $g_k^{(\ell)}(y)$ satisfies

(42) $$\sup_{0 \le s \le t} \bigvee_{j \in \Pi(k)} (g_j^{(\ell)}(y)(s) - x_j(s))^+ \le g_k^{(\ell)}(y)(t) \quad \text{for } t \in [0, T+\delta] .$$

Observe that (42) is a trivial consequence of (40) if $k \notin \underline{S}(M-\ell+1)$. Suppose that $k \in \underline{S}(M-\ell+1)$ and let $t \in [0, T+\delta]$. Since $g^{(\ell)}(y) = y$ on $[0,T]$, it can be assumed that $t \in [T, T+\delta]$. It suffices to show that $j \in \Pi(k)$ satisfies

$$(43) \qquad \sup_{T \le s \le t} (g_j^{(\ell)}(y)(s) - x_j(s))^+ \le g_k^{(\ell)}(y)(t) \quad .$$

Let $s \in [T,t]$. If $g_j^{(\ell)}(y)(s) = y_j(s)$, then

$$g_j^{(\ell)}(y)(s) - x_j(s))^+ = (y_j(s)-x_j(s))^+ \le y_k(t) \le g_k^{(\ell)}(y)(t) \quad .$$

If $g_j^{(\ell)}(y)(s) > y_j(s)$, then there exists $u \in [T,s]$ such that $g_j^{(\ell)}(y)(s) = x_j(u) + y_k(u) - b_j$. Therefore

$$(g_j^{(\ell)}(y)(s) - x_j(s))^+ = (x_j(u) - x_j(s) + y_k(u) - b_j)^+$$

$$\le (\bigwedge_{n=1}^{K} b_n - b_j + y_k(u))^+ \le y_k(u) \le g_k^{(\ell)}(t) \quad .$$

Consequently (42) must be valid. Q.E.D.

(44) Corollary. If $y \in \underline{U}$ satisfies (39)-(41) then $G(y) \in \underline{L}_{T+\delta}$.

The proof of the corollary follows easily from recursive use of Proposition (38), and therefore will be omitted.

Remark. If $y \in \underline{U}$ satisfies

(45) $$\sup_{0 \le s \le t} (x_k(s) - y_\sigma(s) - b_k)^+ \le y_k$$

for $k \in \cup_{n=1}^{\ell-1} \underline{S}(M-n)$ and $t \ge 0$, then for $k \in \cup_{n=1}^{\ell} \underline{S}(M-n)$, $g_k^{(\ell)}(y)$ satisfies (45) for $t \ge 0$.

Also observe that the surjectiveness of σ implies that $\cup_{n=1}^{M-1} \underline{S}(M-n) = \{1, 2, \ldots, K\}$.

Now use the previous remark recursively to deduce the following proposition.

(46) **Proposition.** If $y \in \underline{U}$ then $G_k(y)$ satisifes (45) for $k = 1, \ldots, K$ and $t \ge 0$.

We now introduce the following important result.

(47) **Proposition.** If $y \in \underline{U}$ and y satisfies (11)-(13) on $[0,T]$, then $(G \cdot H)(y)$ satisfies (11)-(13) on $[0, T+\delta]$, where δ satisfies

$$\sup_{T \le s_1 \le s_2 \le T+\delta} \| x(s_1) - x(s_2) \| \le \bigwedge_{n=1}^{K} b_n \,.$$

Proof. Remark (34) and Proposition (36) together guarantee that $H(y)$ satisfies (39)-(41). Corollary (44) guarantees that $(G \cdot H)(y) \in L_{T+\delta}$. Proposition (46) guarantees that $(G \cdot H)(y)$ satisfies (45) on

$[0,\infty)$. Remark (34) implies that $(G\circ H)(y) \in \underline{U}$. Observe that $(G\circ H)(y) \in L_{T+\delta} \cap \underline{U}$ and condition (45) together imply that $(G\circ H)(y)$ satisfies (11)-(13) on $[0, T+\delta]$. Q.E.D.

Let Δ and N be defined as follows:

(48) $$\Delta(\varepsilon) \equiv \sup\{\delta > 0; \sup_{0\le s_1\le s_2\le s_1+\delta\le T} \|(x(s_1)-x(s_2)\| \le \varepsilon\} ,$$

and

(49) $$N \equiv \left[T/\Delta(\bigwedge_{n=1}^{K} b_n)\right] + 1 .$$

<u>Remark</u>. When necessary we will append arguments to $\Delta(\varepsilon)$ and N to indicate their dependence on x and T.

(50) <u>Lemma</u>. $(G\circ H)^N(0)$ satisfies (11)-(13) on $[0,T]$.

<u>Proof</u>. Observe that $0 \in \underline{U}$, and thus Proposition (47) implies that $(G\circ H)(0)$ satisfies (11)-(13) on $[0, \Delta(\wedge_{n=1}^{K} b_n)]$. Now by induction the result will follow. Q.E.D.

Define $\phi(x)$ as follows:

(51) $$\phi(x) \equiv (G\circ H)^N(0) .$$

Due to typographical considerations, we will often use $y = \phi(x)$, except when x is allowed to vary.

We will now show that $\phi(x)$ is the unique element of $\underline{U}$ which satisfies (11)-(13). We begin by showing that $\phi(x)$ is the "least" solution. Observe that if $w \in \underline{U}$ which satisfies (11)-(13) on $[0,T]$, then $(G \cdot H)(w) = w$ on $[0,T]$. Since $G \cdot H$ is a monotonically increasing operator, it follows that

$$(52) \qquad \phi(x) \equiv (G \cdot H)^N(0) \leq (G \cdot H)^N(w) = w \qquad \text{on } [0,T] .$$

The following remark is a direct result of Proposition 2.11.

(53) <u>Remark</u>. If w satisfies conditions (11)-(13) on $[0,T]$ and w_k is increasing at $t \in [0,T]$ then either $z_k^w(t) = b_k$, or there exists $j \in \Pi(k)$ such that $z_j^w(t) = 0$.

For the remainder of this section, let $y = \phi(x)$ and $w \in \underline{U}$, such that w satisfies (11)-(13) on $[0,T]$. The functions y, w satisfy the important order relationship given in the remark below.

(54) <u>Remark</u>. Suppose that for $t \in [0,T]$, $z_k^w(t) = 0$ and $w_\sigma(t) > y_\sigma(t)$. Then $w_k(t) > y_k(t)$.

The next proposition is essential to the proof of the uniqueness of y.

(55) <u>Proposition</u>. If the functions y and w given above satisfy

$$(56) \qquad y = w \quad \text{on } [0,\tau] ,$$

(57) $$\sup_{\tau \le s_1 \le s_2 \le t} |z^w(s_1) - z^w(s_2)| \le \bigwedge_{n=1}^{K} b_n/2 ,$$

and

(58) there exists $k \in \{L+1, \ldots, K\}$ such that $z_k^w(t) = 0$,

then $w_\sigma(t) = y_\sigma(t)$.

Proof. Suppose in contradiction that $w_\sigma(t) > y_\sigma(t)$. Remark (54) implies that $w_k(t) > y_k(t)$. Proposition 2.7 now implies that there exists $t_1 \in (\tau, t]$ such that $w_k(t_1) > y_k(t_1)$ and w_k is increasing at t_1. Remark (53) implies that either $z_k^w(t_1) = b_k$ or there exists $k_1 \in \Pi(k)$ such that $z_{k_1}^w(t_1) = 0$. Conditions (57) and (58) together disallow the former possibility. Therefore, we deduce that there exists $k_1 \in \Pi(k)$ such that $z_{k_1}^w(t_1) = 0$ and $w_k(t_1) > y_k(t_1)$.

Observe that t_1 and k_1 satisfy our hypotheses on k and t. Therefore by inductively using the previous argument we can deduce that there exists $(k_1, k_2, \ldots, k_r)$ and $(t_1, t_2, \ldots, t_r)$ which satisfy:

(59) $z_{k_j}^w(t_j) = 0$ and $w_{k_{j-1}}(t_j) > y_{k_{j-1}}(t_j)$, $j = 1, \ldots, r$,

(60) $k_{j-1} = \sigma(k_j)$ where $k_0 = k$ and $j = 1, \ldots, r$,

(61) $t_j \in (\tau, t_{j-1}]$ where $t_0 = t$ and $j = 1, \ldots, r$,

and

(62) $k_r \in \{1, 2, \ldots, L\}$.

Condition (59) and Remark (54) together imply that $w_{k_r}(t_r) > y_{k_r}(t_r)$. By imitating the primary argument, we deduce that there exists $t_{r+1} \in (\tau, t_r]$ such that $w_{k_r}(t_{r+1}) > y_{k_r}(t_{r+1})$ and that w_{k_r} is increasing at t_{r+1}. Remark (53) implies that $z^w_{k_{r+1}}(t_{r+1}) = b_{k_{r+1}}$. But the equality $z^w_{k_{r+1}}(t_r) = 0$ and condition (57) imply that $z^w_{k_r}(t_{r+1}) \leq z^w_{k_r}(t_r) + b_{k_r}/2 < b_{k_r}$. This contradiction implies that our initial assumption is wrong. Therefore $w_{\sigma(k)}(t) = y_{\sigma(k)}(t)$.

Q.E.D.

(63) Lemma. Let $y = \phi(x)$, and suppose that $w \in \underline{U}$ satisfies (11)-(13) on $[0,T]$. Then $y = w$ on $[0,T]$.

Proof. Set $\tau \equiv \sup\{t \leq T: y = w \text{ on } [0,t]\}$. The continuity of y and w implies that $y = w$ on $[0,\tau]$. Since z^w is continuous there exists $t > \tau$ such that z^w satisfies (57). Now define

(64) $$\ell \equiv \max(\{-1\} \cup \{n \text{ for which there exists } \delta > 0 \text{ such that } w_k(s) = y_k(s) \text{ whenever } s \in [0; \tau+\delta] \text{ and } k \in \bigcup_{m=0}^{n} \underline{S}(M-m)\}).$$

Suppose in contradiction that $\ell < M-1$. By definition of ℓ there exists $k \in \underline{S}(M-\ell-1)$ and $t_0 \in (\tau, t]$ which satisfy $0 \leq y_k(t_0) < w_k(t_0)$. Proposition (2.7) implies that we may assume without loss of generality that w_k is increasing at t_0. Remark (53) implies that either

(65) $$z_k^w(t_0) = b_k \quad ,$$

or

(66) $$z_{k_0}^w(t_0) = 0 \qquad \text{where } k_0 \in \Pi(k) \quad .$$

We will now show that only (66) is possible. If $k = K+1$ then only (65) is possible. If $k < K+1$ then suppose (65) is true. We obtain

(67) $$b_k = z_k^w(t_0) = x_k(t_0)-w_k(t_0)+w_\sigma(t_0)$$

$$= x_k(t_0)-w_k(t_0)+y_\sigma(t_0) < x_k(t_0)-y_k(t_0)+y_\sigma(t_0) \quad .$$

Line (67) implies that $b_k < z_k^y(t_0) \leq b_k$. Therefore we deduce that only (66) is possible. Proposition (55) now implies that $w_k(t_0) = y_k(t_0)$, but this contradicts our assumption that $w_k(t_0) > y_k(t_0)$. Therefore our initial assumption that $\ell < M-1$ must be false, and thus $\ell = M-1$. The definition of ℓ implies that there exists $\delta > 0$ such that $w = y$ on $[0, \tau+\delta]$. The last line contradicts our definition of τ. Therefore $\tau = T$ and thus $w = y$ on $[0,T]$. Q.E.D.

4.2 Continuity of the Mapping

We begin by defining the mapping $\phi\colon C^K \to C^K$ by

(68) $$\phi_k(x) \equiv x_k - \phi_k(x) + \phi_\sigma(x) \; , \qquad \text{for } k = 1, \ldots, K \; .$$

(69) **Theorem.** The mapping $x \to (\psi(x), \phi(x))$ is continuous from C^K to C^{2K+1} with respect to the topology of uniform convergence on finite intervals.

Proof. Since C^{2K+1} is a product space, it suffices to show that both $\psi(x)$ and $\phi(x)$ are continuous. We begin by showing that $\psi(x)$ is continuous. We will now interpret $h^{(\ell)}$, $g^{(\ell)}$, H and G as being mappings from C^{2K+1} to C^{2K+1} in the following way:

$$h^{(\ell)}(y,x) \equiv (h^{(\ell)}(y),\ x) \qquad \text{for } \ell = 1, \ldots, M-1 ,$$

$$g^{(\ell)}(y,x) \equiv (g^{(\ell)}(y),\ x) \qquad \text{for } \ell = 1, \ldots, M-1 ,$$

$$H(y,x) = (H(y),\ x) ,$$

and

$$G(y,x) = (G(y),\ x) ,$$

where $h^{(\ell)}(y)$, $g^{(\ell)}(y)$; $H(y)$ and $G(y)$ are defined by (29), (30), (32) and (33). Since the continuity of mappings is preserved by composition, it follows that H and G are continuous as long as $h^{(\ell)}$ and $g^{(\ell)}$ are continuous maps for $\ell \in \{1, \ldots, M-1\}$. We will now prove that $h^{(\ell)}$ is a continuous map. Let $T > 0$ be given. We need to show that $h^{(\ell)}$ is a continuous mapping on $C^{2K+1}[0,T]$.

Let $(y,x), (y',x') \in C^{2K+1}[0,T]$, and let $k \in \underline{E}(\ell)$;

$$\left|h_k^{(\ell)}(y,x) - h_k^{(\ell)}(y',x')(t)\right|$$

$$= \left|\sup_{0 \le s \le t} \bigvee_{j\in\Pi(k)} (y_j(s)-x_j(s))^+ \vee y_k(s)\right.$$

$$\left. - \sup_{0 \le s \le t} \bigvee_{j\in\Pi(k)} (y_j'(s)-x_j'(s))^+ \vee y_k'(s)\right|$$

$$\le \sup_{0 \le s \le t} \|x(s)-x'(s)\| + 2 \sup_{0 \le s \le t} \|y(s) - y'(s)\|$$

$$\le 3 \sup_{0 \le s \le T} \|(y(s),x(s)) - (y'(s),x'(s))\|$$

$$= 3\|(y,x) - (y',x')\| \quad .$$

Therefore,

(70) $$\left|\sup_{0 \le t \le T} h_k^{(\ell)}(y,x)(t) - h_k^{(\ell)}(y',x')(t)\right| \le 3\|(y,x) - (y',x')\| \quad .$$

It is easily verified that (70) holds for $k \notin \underline{E}(\ell)$. Therefore we obtain

(71) $$\|h^{(\ell)}(y,x) - h^{(\ell)}(y',x')\|$$

$$\equiv \sup_{0 \le t \le T} \left|\bigvee_{k=1}^{K+1} \left|h_k^{(\ell)}(y,x)(t) - h_k^{(\ell)}(y',x')(t)\right|\right.$$

$$\le 3\|(y,x) - (y',x')\| \quad .$$

Line (71) implies that $h^{(\ell)}$ is continuous. In similar fashion it can be shown that $g^{(\ell)}$ is continuous. Therefore G and H must be continuous, and thus $(G\cdot H)^n$ is a continuous map for any fixed n. Moreover, the mapping $x \to (0,x)$ is a continuous mapping from C^K to C^{2K+1}. Now observe that $(\phi(x),x) = (G\cdot H)^{N(x)}(0,x)$. Suppose that $N(x') > N(x)$, then the invariance of $\phi(x)$ implies the equation

$$(G\cdot H)^{N(x')}(0,x) = (G\cdot H)^{N(x')-N(x)}(\phi(x),\, x) = (\phi(x),\, x) \quad .$$

Therefore, it suffices to show that $N(x)$ is a locally bounded function of x. Define $M(x)$ by

$$M(x) \equiv \left[1/\Delta(x, \bigwedge_{n=1}^{K} b_n/2)\right] + 1 \quad .$$

If $x' \in C^K[0,T]$ such that $\sup_{0\le t\le T} \|x(t)-x'(t)\| \le \wedge_{n=1}^{K} b_n/4$, then $N(x') \le M(x)$. To see the last inequality, note that

$$\sup_{0 \le s_1 \le s_2 \le s_1+\Delta \le T} \|x'(s_1) - x'(s_2)\|$$

$$\le \sup_{0 \le s_1 \le s_2 \le s_1+\Delta \le T} \Big[\|x'(s_1) - x(s_1)\|$$

$$+ \|x(s_1) - x(s_2)\| + \|x(s_2) - x'(s_2)\|\Big]$$

$$\le \bigwedge_{n=1}^{K} b_n\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right) = \bigwedge_{n=1}^{K} b_n \ ,$$

where $\Delta = \Delta(x, \wedge_{n=1}^{K} b_n/2)$.

Therefore by definition of Δ we find that $\Delta(x', \wedge_{n=1}^{K} b_n) \geq \Delta(x, \wedge_{n=1}^{K} b_n/2)$. The definition of N and M together imply that $N(x') \leq M(x)$. We now see that the mapping $(\phi(x'), x') = (G\cdot H)^{M(x)}(0,x')$ is locally a <u>fixed</u> power of $G\cdot H$. It is now clear that $(\phi(x), x)$ and $\phi(x)$ are continuous. Since $\phi(x) = (\phi(x), x)\cdot A$, where A is a $2K+1$ by K matrix, it also follows that ϕ is continuous. Q.E.D.

4.3 Additional Properties

(72) <u>Lemma</u>. Fix $x \in C_S$ and $T > 0$. Let $y = \phi(x)$ and $z = \phi(x)$ as before. Define $x^*(t) \equiv z(T) + x(T+t) - x(T)$, $y^*(t) = y(T+t)-y(T)$, and $z^*(t) = z(T+t)$. Then $y^* = \phi(x^*)$ and $z^* = \phi(x^*)$.

<u>Proof</u>. If y^* satisfies (11)-(13) for x^* then the equality $y^* = \phi(x^*)$ will follow from Lemma (63). We begin this task by showing that y^* satisfies (12) for x^*. Let $k \in \{L+1, \ldots, K\}$ and $t > 0$ be given. Suppose that $y_k^*(t) > 0$, then $y_k(T+t) > y_k(T)$. From (12) it follows that

$y_k^*(T+t)-y_k^*(T)$

$$= \sup_{T \le s \le T+t} \bigvee_{j\in\Pi(k)} (y_j(s)-x_j(s)-y_k(T))^+ \vee (x_k(s)+y_\sigma(s)-y_k(T)-b_k)^+$$

$$= \sup_{T \le s \le T+t} \bigvee_{j\in\Pi(k)} \Big[(y_j(s)-y_j(T)-x_j(s)+x_j(T)+y_j(T)-x_j(T)-y_k(T))^+$$

$$\vee (x_k(s)-x_k(T)+y_\sigma(s)-y_\sigma(T)+x_k(T)+y_\sigma(T)-y_k(T)-b_k)^+\Big]$$

$$= \sup_{T \le s \le T+t} \bigvee_{j\in\Pi(k)} (y_j^*(s-T)-x_j^*(s-T))^+ \vee (x_k^*(s-T)+y_\sigma^*(s-T)-b_k)^+ \ .$$

The last equality is obtained from the previous expression by using the definitions of y^* and x^*. If $u = s-T$ is substituted for s in the last line we deduce *that*

$$y_k(T+t)-y_k(T) = \sup_{0 \le u \le t} \bigvee_{j\in\Pi(k)} (y_j^*(u)-x_j^*(u))^+ \vee (x_k^*(u)+y_\sigma^*(u)-b_k)^+$$

The last line shows that y_k^* satisfies (12) whenever $y_k^* > 0$. Let us now suppose that $y_k^*(t) = 0$. Then $y_k(T+t) = y_k(T)$, and thus (12) implies

$$y_k(T+t) \ge \sup_{T \le s \le T+t} \bigvee_{j\in\Pi(k)} (y_j(s)-x_j(s))^+ \vee (x_k(s)+y_\sigma(s)-b_k)^+$$

$$= \sup_{T \le s \le T+t} \bigvee_{j\in\Pi(k)} (y_j^*(s-T)-x_j^*(s-T)+y_k(T))^+$$

$$\vee (x_k^*(s-T)+y_\sigma^*(T-t)+y_k(T)-b_k)^+ \ .$$

The last equality follows again by simple substitution. Now substitute $u = s-T$ to obtain

$$(73) \quad y_k(T+t) \geq \sup_{0 \leq u \leq t} \bigvee_{j\in\Pi(k)} (y_j^*(u)-x_j^*(u)+y_k(T))^+$$

$$\vee \; (x_k^*(u)+y_\sigma^*(u)+y_k(T)-b_k)^+ \; .$$

Since $y_k(T+t) = y_k(T)$, it follows from (73) that

$$(74) \quad 0 = \sup_{0 \leq u \leq t} \bigvee_{j\in\Pi(k)} (y_j^*(u)-x_j^*(u))^+ \vee (x_k^*(u)+y_\sigma^*(u)-b_k)^+ \; .$$

Therefore, it now follows that y^* satisfies (12) for x^*. In precisely the same fashion it can be shown that y^* satisfies (11) and (13) for x^*. Thus we have shown that $y^* = \phi(x^*)$. It remains to show that $z^* = \phi(x^*)$. Let $k \in \{1, 2, \ldots, K\}$ be given,

$$\begin{aligned}
\phi_k(x^*)(t) &\equiv x_k^*(t)-\phi_k(x^*)(t) + \phi_\sigma(x^*)(t) = x_k^*(t)-y_k^*(t)+y_\sigma^*(t) \\
&= z_k(T)+x_k(T+t)-x_k(T)-y_k(T+t)+y_k(T)+y_\sigma(T+t)-y_\sigma(T) \\
&= x_k(T+t)-y_k(T+t)+y_\sigma(T+t)+z_k(T)-z_k(T) \\
&= z_k(T+t) \equiv z_k^*(t) \quad .
\end{aligned}$$

Therefore, it now follows that $z^* = \phi(x^*)$. Q.E.D.

In the preceding b was taken to be fixed; in this part b will be allowed to vary. Let $x \in C_S$ and $b = (b_1, \ldots, b_K)$ where $b_k > 0$. Define $\psi(x,b)$ to be the function determined by (11)-(13). Define $\phi(x,b)$ in terms of x and $\psi(x,b)$ according to (68).

(75) <u>Lemma</u>. The functions $\psi(x,b)$ and $\phi(x,b)$ satisfy

$$\psi\left(\frac{x(r\cdot)}{r^{1/2}}, b\right)(\cdot) = r^{-1/2}\, \psi(x, r^{1/2} b)(r\cdot) \quad , \tag{76}$$

and

$$\phi\left(\frac{x(r\cdot)}{r^{1/2}}, b\right)(\cdot) = r^{-1/2}\, \phi(x, r^{1/2} b)(r\cdot) \quad , \text{ where } r > 0. \tag{77}$$

<u>Proof</u>. Set $w \equiv \psi(x, r^{1/2} b)(r\cdot)$. We will show that $r^{-1/2} w$ satisfies (11)-(13) for $x(r\cdot)\, r^{-1/2}$ and then apply Lemma 63 to deduce (76). Let $k \in \{L+1, \ldots, K\}$ and $t > 0$. The function w satisfies by definition:

$$w_k(t) = \sup_{0 \le s \le rt} \; \bigvee_{j \in \Pi(k)} \left(w_j\left(\frac{s}{r}\right) - x_j(s)\right)^+ \vee \left(x_k(s) + w_\sigma\left(\frac{s}{r}\right) - r^{1/2} b_k\right)^+ . \tag{78}$$

Substitute $u = s/r$ into (78) and simplify to obtain

$$(79) \qquad w_k(t) = \sup_{0 \le u \le t} \bigvee_{j \in \Pi(k)} (w_j(u) - x_j(ur))^+$$

$$\vee \left(x_k(ur) + w_\sigma(u) - r^{1/2}\, b_k\right)^+ .$$

Divide equation (79) by $r^{1/2}$ to deduce

$$(80) \quad r^{-1/2}\, w_k(t) = \sup_{0 \le u \le t} \bigvee_{j \in \Pi(k)} (w_j(u)\, r^{-1/2} - x_j(ur)\, r^{-1/2})^+$$

$$\vee\ (x_k(ur)\, r^{-1/2} + w_\sigma(u)\, r^{-1/2} - b_k)^+ .$$

Equation (80) implies that $r^{-1/2}\, w$ satisfies (12) for $x(r\cdot)\, r^{-1/2}$ and b. Similarly it can be shown that $r^{-1/2}\, w$ satisfies (11) and (13) for $x(r\cdot)\, r^{-1/2}$ and b. It now follows that (76) is valid. To prove (77), begin by observing that

$$\phi_k(x(r\cdot)\, r^{-1/2}, b) \equiv \frac{x(r\cdot)}{r^{1/2}} - \phi_k(x(r\cdot)\, r^{-1/2}, b) + \phi_\sigma(x(r\cdot)\, r^{-1/2}, b) .$$

Use (76) to deduce that

$$\phi_k(x(r\cdot)\, r^{-1/2}, b)$$

$$= x(r\cdot)\, r^{-1/2} - r^{-1/2}\, \phi_k(x, r^{1/2} b)(r\cdot) + r^{-1/2}\, \phi_\sigma(x, r^{1/2} b)(r\cdot)$$

$$= r^{-1/2}\, [x(r\cdot) - \phi_k(x, r^{1/2}\, b)(r\cdot) + \phi_\sigma(x, r^{1/2} b)(r\cdot)]$$

$$= r^{-1/2}\, \phi_k(x, r^{1/2} b)(r\cdot) .$$

Therefore (77) holds as well. Q.E.D.

(81) Proposition. Let $x, x' \in C_S$, and suppose $x = x'$ on $[0,t]$, then $\psi(x) = \psi(x')$ and $\phi(x) = \phi(x')$ on $[0,t]$.

The proof of Proposition 81 can be easily derived from equation (51) and the definition of $G \bullet H$. Therefore we omit the proof.

4.4 A Bound for the Boundary Process

Let $w \in C^{K+1}$, $x \in C^K$, and $T > 0$ be given.

(82) Proposition. The mapping $G \bullet H$ satisfies

(83) $$\|(G \bullet H)(w,x)\| \equiv \sup_{0 \le t \le T} \|(G \bullet H)(w,x)(t)\| \le 2M(\|x\| + \|w\|) .$$

Proof. Observe that $g^{(\ell)}$ and $h^{(\ell)}$ satisfy

(84) $$\|h^{(\ell)}(w,x)\| \le \|w\| + \|x\| , \quad \text{for } \ell = 1, 2, \ldots, M-1 ,$$

and

(85) $$\|g^{(\ell)}(w,x)\| \le \|w\| + \|x\| , \quad \text{for } \ell = 1, 2, \ldots, M-1 .$$

Since $G \bullet H \equiv g^{(M-1)} \bullet \cdots \bullet g^{(1)} \bullet h^{(M-1)} \bullet \cdots \bullet h^{(1)}$, line (83) now follows from (84) and (85) by recursion.

(86) Corollary. The mapping $x \to (G \bullet H)(0,x)$ satisfies

(87) $$\|(G\circ H)(0,x)\| \le 2M(\|x\|) \quad .$$

(88) Lemma. Let $x \in C^K$ be given. Then

(89) $$\|\phi(x)(T)\| \le N(x) \cdot 4M\Big(\bigvee_{n=1}^{K} b_n\Big) \; ,$$

where $N \equiv N(x)$ is defined by (49).

Proof. The proof proceeds by induction on N. If $N = 1$ then (49) implies that

(90) $$\sup_{0 \le s \le T} \|x(0) - x(s)\| \le \bigwedge_{n=1}^{K} b_n \; .$$

Since $x(0) \in S$, it follows from (90) that

(91) $$\|x\| \le \|x(0)\| + \bigwedge_{n=1}^{K} b_n \le 2 \bigvee_{n=1}^{K} b_n \; .$$

Line (91) and Corollary 86 together imply (89).

Now suppose by induction that (89) holds for all $z \in C^K$ such that $N(z,T) \le m$ and all $T > 0$. Let $x \in C^K$ such that $N(x,T) = m+1$. Define $x^* \in C^K$ as follows,

(92) $$x^*(t) \equiv \phi(x)(\Delta) + x(t+\Delta) - x(\Delta) ,$$

where $\Delta \equiv \Delta(x, \wedge_{n=1}^{K} b_n)$ is defined by (48).

The definition of N implies that $N(x^*, T-\Delta) \leq m$. Therefore

(93) $$\|\phi(x^*)(T-\Delta)\| \leq (N(x)-1)\ 4M\left(\bigvee_{n=1}^{K} b_n\right) .$$

Lemma 72 implies that

(94) $$\phi(x)(T) = \phi(x)(\Delta) + \phi(x^*)(T-\Delta) .$$

Lines (93) and (94) and the induction hypothesis together imply (89).

Q.E.D.

(95) Corollary. Let $x \in C^K$ satisfy $x(0) = 0$. Then

(96) $$\sup_{x_0 \in S} \|\phi(x_0+x)(T)\| \leq N(x)\ 4M\left(\bigvee_{n=1}^{K} b_n\right) .$$

Proof. Observe that $N(x) = N(x_0+x)$ for any $x_0 \in S$. Line (96) now follows from Lemma 88. Q.E.D.

CHAPTER 5

REFLECTED BROWNIAN MOTION

This chapter is devoted to study of the K-dimensional stochastic process $Z \equiv \phi(X)$, where ϕ is the reflection mapping of Chapter 4 and X is a K-dimensional Brownian Motion (with arbitrary drift and covariance matrix). It will be shown that Z is a diffusion process, and some of its properties will be explored. In particular, it will be shown that Z has a unique stationary distribution, and an analytical characterization of that distribution will be developed.

5.1 The Diffusion Property

This section begins with the introduction of notation necessary for defining a diffusion process on R^K.

Let I be a rectangle on R^K and let C^I be the space of continuous functions from $[0,\infty)$ to I. Let $(\Omega,\underline{F})$ be a measurable space and suppose $Z\colon \Omega \to C^I$. Denote $Z(\omega)(t)$ by $Z(t)$. Let $\underline{F}_t$ be a filtration of σ-fields such that

$$(1) \qquad Z(t) \in \underline{F}_t \subset \underline{F}\,, \qquad \text{for} \quad t \geq 0\,.$$

For each $x \in I$, let P^x be a probability measure on $(\Omega,\underline{F})$. Let $\tau\colon \Omega \to [0,\infty]$. We say that τ is a Markov time relative to $\{\underline{F}_t, t \geq 0\}$ and $\{P^x, x \in I\}$ if

(2) $$\{\tau \leq t\} \in \underline{F}_t \qquad \text{for} \quad t \geq 0 \quad ,$$

and

(3) $$P^x(\tau < \infty) = 1 \qquad \text{for} \quad x \in I \; .$$

Let $\underline{F}_\tau$ be the pre-τ field, i.e.,

(4) $$\underline{F}_\tau \equiv \{A \in \underline{F}\colon A \cap \{\tau \leq t\} \in \underline{F}_t \quad \text{for} \quad t \geq 0\} \quad .$$

We will call $\{Z(t), t \geq 0\}$ a <u>diffusion</u> on I if

(5) $$P^x(Z(0) = x) = 1 \qquad \text{for all} \quad x \in I \; ,$$

and for all bounded, continuous functions on I and all Markov times τ

(6) $$E^x[f(Z(t+\tau)|\underline{F}_\tau] = E^{Z(\tau)}[f(Z(t))] \qquad \text{a.s.} \quad P^x \; .$$

For the remainder of this chapter let $\Omega = C^K$. Denote a generic element of Ω by $\omega = ((\omega_1(t), \ldots, \omega_K(t)); \; t \geq 0)$. Let X be the coordinate process on Ω, i.e.,

(7) $$X(\omega,t) = \omega(t) \qquad \text{for} \quad t \geq 0 \quad \text{and} \quad \omega \in \Omega \quad .$$

Denote $X(\omega,t)$ by $X(t)$.

Let $\underline{F}$ be the Borel field on Ω generated by the topology of uniform convergence on compact intervals. Let $\underline{F}_t \equiv \wedge_{\varepsilon>0} \sigma\{X(s)\colon$

$0 \leq s \leq t+\varepsilon\}$. For each $x \in R^K$, let P^x be the probability measure on $(\Omega, \underline{F})$ under which X is a Brownian motion with drift vector μ, positive definite covariance matrix A, and starting state x. Define $Z = \phi(X)$ and $Y = \psi(X)$ with the reflection map (ψ, ϕ) given in Chapter 4.

Let τ be a Markov time; define X^* and Y^* as follows:

$$(8) \qquad X^*(t) \equiv Z(\tau) + X(\tau+t) - X(\tau) \qquad \text{for} \quad t \geq 0 ,$$

and

$$(9) \qquad Y^*(t) \equiv Y(\tau+t) - Y(\tau) \qquad \text{for} \quad t \geq 0 .$$

We will now prove that $\{Z(t), t \geq 0\}$ is a diffusion on S. But first we need to prove several preliminary propositions.

(10) __Proposition.__ For every $t \geq 0$, $(Y(t), Z(t)) \in \underline{F}_t$.

__Proof.__ This measurability property follows easily from the measurability preserving property of $G \circ H$ (see Lemma 4.50). The measurability of $G \circ H$ in turn follows from Proposition 2.16.

(11) __Proposition.__ Every Markov time τ and every bounded random variable $W \in \sigma\{Z(\tau), X(\tau+t) - X(\tau), t \geq 0\}$ together satisfy

$$(12) \qquad E^x[W | \underline{F}_\tau] = E^x[W | Z(\tau)] \qquad \text{a.s. } P^x \text{ for } x \in S .$$

__Proof.__ By the usual monotone class argument it suffices to show that

$$W = f_0(Z(\tau)) \cdot f_1(X(\tau+t_1)-X(\tau)) \cdots f_n(X(\tau+t_n) - X(\tau))$$

satisfies (12), where f_j, $j = 1, \ldots, n$, is a bounded Borel function and $0 < t_1 < t_2 < \cdots < t_n$. Observe that if $Z(\tau) \in \underline{F}_\tau$, it follows that

$$(13) \qquad E^x[W|\underline{F}_\tau] = f_0(Z(\tau))\, E^x[f_1(X(\tau+t_1)-X(\tau)) \cdots f_n(X(\tau+t_n)-X(\tau))|\underline{F}_\tau] \,.$$

By the strong Markov property of Brownian motion it follows from (13) that

$$(14) \qquad E^x[W|\underline{F}_\tau] = f_0(Z(\tau))\, E^x[f_1(X(\tau+t_1)-X(\tau)) \cdots f_n(X(\tau+t_n)-X(\tau))] \,.$$

Notice that the last expression is a function that belongs to $\sigma(Z(\tau))$ and therefore (12) must hold. It now suffices to show that $Z(\tau) \in \underline{F}_\tau$. To show this, begin by observing that if τ is countable the result is easy to prove. Notice that $\tau_n = [n\tau+1]/n$ is a Markov time. Therefore $Z(\tau_n) \in \underline{F}_{\tau_n}$. But $\underline{F}_\tau = \underline{F}_{\tau+} = \wedge_{n=1} \underline{F}_{\tau_n}$. Since Z is continuous we have

$$z(\tau) = \lim_{n\to\infty} Z(\tau_n) \in \bigwedge_{n=1} \underline{F}_{\tau_n} = \underline{F}_\tau \,. \qquad \text{Q.E.D.}$$

(15) Lemma.

$$E^x[f(Z(\tau+t))|\underline{F}_t] = E^{Z(\tau)}[f(Z(t))] \qquad \text{a.s.} \quad P^x$$

and

$$E^x[g(Y(\tau+t) - Y(\tau))|\underline{F}_\tau] = E^{Z(\tau)}[g(Y(\tau+t) - Y(\tau))] \quad \text{a.s.} \quad P^x,$$

where f and g are bounded, continuous functions.

Proof. Line (4.9) and Proposition 10 together imply that

(16) $$f(Z(\tau+t)),\ g(Y(\tau+t) - Y(\tau)) \in \sigma\{X^*(t),\ t \geq 0\} \ .$$

Line 16 and Proposition 11 together imply (15).

(17) Theorem. The process $\{Z(t),\ t \geq 0\}$ is a diffusion on S.

Proof. Observe that (5) follows from the identity $Z(0) = X(0)$. Condition (6) was verified in Lemma 15.

5.2 Ergodicity

In this section we will prove that Z satisfies the conditions of Theorem 2.59.

(18) Proposition. Let W be the standard K-dimensional Brownian motion on $[0,T]$ with $W(0) = 0$. Suppose B is an open set of the form:

(19) $$B \equiv \{y: \|y-x\| < \varepsilon\} , \quad \text{where } x \text{ satisfies}$$

(20) $$x(t) = ct \quad , \qquad t \in [0,T] \text{ and } c \in R^K .$$

Then $P(W \in B) > 0$.

Proof.

$$P(W \in B) = P\{ \sup_{0 \le t \le T} \|W(t) - ct\| < \varepsilon\}$$

$$= \prod_{k=1}^{K} P\{ \sup_{0 \le t \le T} \left|W_k(t) - c_k t\right| < \varepsilon\} .$$

Observe that if $c = 0$ then the result follows from elementary properties of 1-dimensional Brownian motion. For $c_k \neq 0$, the usual likelihood ratio argument (for example, see (2.53) above) will show that

(21) $$P(\sup_{0 \le t \le T} \left|W_k(t) - c_k t\right| < \varepsilon) > 0 . \qquad \text{Q.E.D.}$$

(22) Corollary. If X is any fully K-dimensional, Brownian motion with $X(0) = 0$ then $P(X \in B) > 0$.

Proof. We can represent X as

$$X = AW + \mu t \tag{23}$$

where A is a $K \times K$ non-singular matrix and μ is a K-dimensional vector. Observe that there exists $\delta > 0$ such that

$$\{\omega: \sup_{0 \le t \le T} \|W(t) - A^{-1}(c-\mu)t\| < \delta\} \subset \{X \in B\} .$$

Therefore $P(X \in B) > 0$ follows from Proposition 18.

(24) Proposition. The process Z satisfies property (2.60).

Proof. Let $z, y \in S$, and $T, \varepsilon > 0$ be given. Define Q a subset of C^K as follows:

$$Q \equiv \{x \in C^K: \|x(T) - y\| < \varepsilon\} .$$

Define

$$B \equiv \{\omega: \|Z(T)-y\| < \varepsilon\} = \{\omega: X \in \phi^{-1}(Q)\} .$$

Since ϕ is continuous and Q is open, it follows that $\phi^{-1}(Q)$ is open. Define

$$\phi^{-1}(Q)-z \equiv \{x(t)-z: \quad x(t) \in \phi^{-1}(Q)\} .$$

Clearly $\phi^{-1}(Q)-z$ is open. Furthermore, the function $(1 - t/T)z + (t/T)y$ belongs to $\phi^{-1}(Q)$. Therefore $x(t) \equiv (t/T)(y-z)$ belongs to $\phi^{-1}(Q)-z$. Corollary 22 now implies

$$(25) \qquad P^z(Z(T) \in (y-\varepsilon,\ y+\varepsilon)) = P^0(X \in \phi^{-1}(Q)-z) > 0 \ .$$

The general result follows easily from (25). Q.E.D.

(26) <u>Proposition</u>. The process Z satisfies property (2.61).

<u>Proof</u>. Let $f \in C(S)$ and $x \in S$ be given. Observe that

$$(27) \qquad E^y(f(Z(s)) = E^0[f(\phi(y+X)(s))] \ .$$

Since ϕ is continuous it follows that

$$(28) \qquad f(\phi(y+X)(s)) \to f(\phi(x+X)(t)) \quad \text{as} \quad s \to t \quad \text{and} \quad y \to x \ .$$

Since f is continuous and S compact we have

$$(29) \qquad \left|f(\phi(y+X)(s))\right| \le \|f\| \ .$$

Lines (27), (28), (29) and the Dominated Convergence Theorem together imply that

$$(30) \qquad \lim_{s\to t,y\to x} E^y(f(Z(s)) = E^x(f(Z(t)) \ . \qquad \text{Q.E.D.}$$

(31) <u>Proposition</u>. Every state of the Brownian motion X is diffusion-like.

Proof. We begin by observing that the spatial homogeneity of the X process allows us to consider only the 0 state. There exist a nonsingular $K \times K$ matrix D and K-dimensional vector c such that $W(t) = DX(t)$, where W is standard Brownian motion with drift c.

Define $B_r \equiv \{D^{-1}x: \|x\| \leq r\}$ and τ by

(32) $$\tau \equiv \inf\{s; X(s) \in \partial B_r\} .$$

Let Q be a measurable subset of $R_+ \times \partial B_r$, and define $Q^* \equiv \{(s,Dz)$ where $(s,z) \in Q\}$. Thus it follows that

(33) $$P^x((\tau, X(\tau)) \in Q) = P^x((\tau, W(\tau)) \in Q^*) .$$

Line (33) and Proposition 2.50 together imply that

(34) $$\lim_{x \to 0} \sup_{Q \in \underline{Q}} \left|P^x((\tau, X(\tau)) \in Q)) - P^0((\tau, X(\tau)) \in Q)\right| = 0 ,$$

where $\underline{Q}$ is the class of Borel measurable subsets of $R_+ \times \partial B_r$. Finally, $P^x(\tau < \infty) = 1$ because Brownian motion always escapes bounded neighborhoods.

(35) Proposition. The diffusion-like states of Z are dense in S.

Proof. Since Z behaves like X in the interior of S, the proposition is implied by Proposition 31. Q.E.D.

Proposition 2.59 now implies that Z is an ergodic process.

5.3 A Change of Variable Formula

It will be convenient to represent X in the form $X(t) = W(t)+\mu t$ where $W \equiv \{W(t), t \geq 0\}$ is a Brownian motion with covariance A, zero drift and $W(0) = X(0) = Z(0)$. Then we have

$$(36) \qquad Z_k(t) = W_k(t) + M_k(t) ,$$

where

$$(37) \qquad M_k(t) \equiv \mu_k t - Y_k(t) + Y_\sigma(t) , \qquad t \geq 0 ,$$

for $k = 1, \ldots, K$. Observe that W_k is a martingale over $\{\underline{F}_t\}$ and that M_k is a continuous adapted process of bounded variation. Thus each Z_k is a continuous semimartingale, and one can develop the analytical theory of vector process Z from the following version of Ito's formula: For twice differentiable functions on R^k define the (constant coefficient) differential operators

$$(38) \qquad L \equiv \sum_{i=1}^{K} \sum_{j=1}^{K} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{K} \mu_i \frac{\partial f}{\partial X_i} ,$$

and

$$(39) \qquad D_k \equiv \sum_{j\in\Pi(k)} \frac{\partial}{\partial X_j} - \frac{\partial}{\partial X_k} \qquad \text{for } k = 1, \ldots, K ,$$

and

$$(40) \qquad D_{K+1} \equiv \sum_{j\in\Pi(K+1)} \frac{\partial}{\partial X_j} .$$

Observe that D_k is the directional derivative in the direction of reflection at the boundary $\{Z_k = b_k\} \cup \{\wedge_{j\in\Pi(k)} Z_j = 0\}$. For

$k = K+1$, D_{K+1} is the directional derivative in the direction of reflection at the boundary $\{\wedge_{j\in\Pi(K+1)} Z_j = 0\}$. Let $\underline{S}$ (for smooth) be the class of functions $f(t,x)$ that are continuously differentiable in $t \geq 0$ and twice continuously differentiable in $x \in R^K$.

(41) Theorem. If $f \in \underline{S}$ then

(42) $$f(t,Z(t)) - f(0,Z(0)) = \int_0^t \left[\frac{\partial f}{\partial s}(s,Z(s)) + Lf(s,Z(s))\right] ds$$

$$+ \sum_{k=1}^{K} \int_0^t \frac{\partial f}{\partial x_k}(s,Z(s))\, dW_k(s)$$

$$+ \sum_{k=1}^{K+1} \int_0^t D_k f(s,Z(s))\, dY_k(s)$$

for all $t \geq 0$. Here the integrals involving $dW_k(s)$ are of the Ito type, and those involving $dY_k(s)$ are defined path by path as ordinary Riemann-Stieltjes integrals.

Proof. By making minor changes in the proof of the Kunita-Watanabe [9] change of variable formula it can be shown that the following equations are valid.

$$(43)\quad f(t,Z(t)) = f(0,Z(0)) + \int_0^t \frac{\partial}{\partial s} f(s, Z(s))\, ds$$

$$+ \sum_{k=1}^{K} \int_0^t \frac{\partial}{\partial X_k} f(s,Z(s))\, dZ_k(s) + \sum_{k=1}^{K} \int_0^t \mu_k \frac{\partial}{\partial X_k} f(s,Z(s))\, ds$$

$$+ \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \int_0^t \frac{\partial^2}{\partial X_i \partial X_j} f(s,Z(s))\, dZ_i(s)\, dZ_j(s) ,$$

where the differentials are computed from (36) and (37) in the obvious way and

$$(44)\qquad dZ_i(t)\, dZ_j(t) = a_{ij}\, dt$$

by convention.

Equation (42) is now obtained from (43) and (44) by simply collecting terms.

(45) Definition. Define S_k as follows:

$$(46)\qquad S_k \equiv \{x \in S: x_k = b_k\} \qquad \text{for } k = 1, \ldots, L$$

$$(47)\qquad S_k \equiv \{x \in S: (\bigwedge_{j\in\Pi(k)} x_j)(x_k - b_k) = 0\} \quad \text{for } k = L+1, \ldots, K$$

$$(48)\qquad S_{K+1} \equiv \{x \in S: \bigwedge_{j\in\Pi(K+1)} x_j = 0\} .$$

(49) Corollary. Let $f \in \underline{S}$ and suppose in addition that

(50) $$D_k f(t,x) = 0 \qquad \text{for } x \in S_k \text{ and } k = 1, \ldots, K+1$$

over the interval $0 \le t \le T$. Define

(51) $$M(t) \equiv f(t,Z(t)) - \int_0^t \left[\frac{\partial}{\partial s} f(s,Z(s)) + Lf(s,Z(s))\right] ds .$$

Then $\{(M(t), \underline{F}_t);\ 0 \le t \le T\}$ is a martingale.

Proof. If (50) holds, then each term in the final summation on the right hand side of (42) vanishes, because $Y_k(\cdot)$ increases only at those times where $Z(t) \in S_k$. Thus Theorem 41 implies that

(52) $$M(t) - M(0) = \sum_{k=1}^{K} \int_0^t \frac{\partial}{\partial x_k} f(s,Z(s))\, dW_k(s) , \qquad 0 \le t \le T.$$

Observe that the continuous functions $\partial/\partial x_1 f, \ldots, \partial/\partial x_K f$ are bounded on the compactum $[0,T] \times S$. Consequently the integrands of the Ito integrals of (52) are bounded, so the right hand side of (51) must be a martingale.

(53) Corollary. Let $h: S \to R$ be continuous. Suppose that $f \in \underline{S}$ and that f satisfies condition (50). If

(54) $$\frac{\partial}{\partial t} f(t,x) + Lf(t,x) = 0 \qquad \text{for } x \in S \text{ and } t \in [0,T] ,$$

and if

$$(55) \qquad f(T,x) = h(x) \qquad \text{for} \quad x \in S$$

then

$$(56) \qquad E[h(Z(T))] = E[f(0, Z(0)] \quad .$$

The proof follows immediately from Corollary 49.

5.4 The Stationary Equation

Let Π be the stationary distribution for Z, meaning that

$$(57) \qquad \Pi(A) = \int_A \Pi(dx)\ P_t(x,A) \qquad \text{for all} \quad t > 0$$

and A a Borel measurable subset of S, and where $P_t(\cdot,\cdot)$ is the transition distribution for Z.

Let E^* be the expectation operator associated with Π. For $k = 1, 2, \ldots, K+1$ define ν_k a finite measure on S as

$$(58) \qquad \nu_k\ (A) \equiv E^* \int_0^1 I_A(Z(t))\ dY_k(t) \quad .$$

Later in this section we will prove that ν_k is indeed a finite measure on S.

(59) Theorem. The measures ν_k, $k = 1, 2, \ldots, K+1$, and Π satisfy the following equation

$$(60) \qquad 0 = \int_S Lf(x)\ \Pi(dx) + \sum_{k=1}^{K+1} \int_{S_k} D_k f(x)\ \nu_k\ (dx) \qquad \text{for} \quad f \in \underline{S} \quad .$$

The remainder of this section is devoted to proving Theorem 59.

(61) Proposition. Let $t > 0$ be given. Then there exists $\alpha_t < \infty$ such that

$$(62) \qquad E^x(\|Y(t)\|) \le \alpha_t \quad \text{for } x \in S .$$

Proof. Define $\xi(s) = X(s)-X(0)$. Observe that $\xi(t)$ is independent of $X(0)$ and that ξ is a Brownian motion with drift μ, covariance matrix A and $\xi(0) \equiv 0$. Observe that $N(X,t) = N(\xi,t)$.

Corollary 4.95 implies that

$$(63) \qquad E^x(\|Y(t)\|) \le E^x(N(\xi,t)) \cdot 4M\left(\bigvee_{n=1}^{K} b_n\right) ,$$

where $N(\xi,t)$ is defined by (4.49), and M is the number of stages. Elementary properties of Brownian motion imply that $E^0(N(\xi,t)) < \infty$. Since ξ is independent of $X(0)$ it follows that

$$(64) \qquad E^x(N(\xi,t)) = E^0(N(\xi,t)) < \infty .$$

Lines (63) and (64) together imply (62).

(65) Corollary. Every $k \in \{1, 2, \ldots, K+1\}$ satisfies

$$(66) \qquad E^*(Y_k(t)) < \infty \quad \text{for } t \ge 0 .$$

The proof of (66) is a direct consequence of (62).

Let $k \in \{1, 2, \ldots, K+1\}$ and $t > 0$ be given. For $f \in C(S)$ define $\lambda_t(f)$ as follows:

(67) $$\lambda_t(f) \equiv E^*\left[\int_0^t f(Z(s))\, dY_k(s)\right] .$$

(68) Proposition. The operator λ_t is bounded, linear and positive.

Proof. The linearity and positivity of λ_t follows easily from the linearity and positivity of E^*. If $\|f\| \leq 1$ then

(69) $$\left|\lambda_t(f)\right| \leq E^*\left[\int_0^t \|f\|\, dY_k(s)\right] \leq E^*\, Y_k(t) \leq \alpha_t . \qquad \text{Q.E.D.}$$

(70) Corollary. The operator λ_t has the following representation:

(71) $$\lambda_t(f) = \int_S f(x)\, \hat{\lambda}_t(dx) \quad ,$$

where $\hat{\lambda}_t$ is a finite measure on the Borel sets of S.

Proof. Observe that S is a compact Hausdorff space, and that λ_t is a bounded linear functional on $C(S)$. Therefore the Riesz Representation Theorem (see Royden [12]) implies that there exists a

post-t field. The penultimate equality follows from the stationarity of the Z process and (4.9). Q.E.D.

(74) Proposition. Every $t \geq 0$ and $f \in C(S)$ together satisfy

$$(75) \qquad \lambda_t(f) \equiv E^*\left[\int_0^t f(Z(s))\, dY_k(s)\right] = t \int_S f(x)\, \nu_k(dx) \; .$$

Proof. Let $f \in C(S)$ be given. The function $g(t) \equiv \lambda_t(f)$ is linear and continuous, therefore elementary function theory implies that

$$(76) \qquad \lambda_t(f) = g(t) = tg(1) = t\lambda_1(f) = t \int_S f(x)\, \lambda_1(dx) \; .$$

Finally, ν_k equals λ_1 by definition. Consequently (75) must hold.

(77) Proposition. The measure ν_k is supported by S_k.

Proof. Observe that lines (4.5)-(4.7) imply that

$$(78) \qquad I_S = I_{S_k} \quad \text{a.s. } Y_k(\omega) \text{ for every } \omega \in \Omega \; .$$

Therefore,

$$(79) \qquad E^* \int_0^1 I_{S_k}(Z(t))\, dY_k(t) = E^* \int_0^1 I_S(Z(t))\, dY_k(t) \; ,$$

finite signed measure $\hat{\lambda}_t$ which satisfies (71). Since λ_t is a positive operator, it follows that $\hat{\lambda}_t$ must be a positive measure.

Q.E.D.

In the following we will let the symbol λ_t represent both the operator λ_t and the measure $\hat{\lambda}_t$.

(72) <u>Proposition</u>. Every $t,s > 0$ and $f \in C(S)$ satisfy

(73) $$\lambda_{t+s}(f) = \lambda_t(f) + \lambda_s(f) \quad .$$

<u>Proof</u>.

$$\lambda_{t+s}(f) \equiv E^*\Big[\int_0^{t+s} f(Z(u))\, dY_k(u)\Big]$$

$$= E^*\Big[\int_0^{t} f(Z(u))\, dY_k(u)\Big] + E^*\Big[\int_t^{t+s} f(Z(u))\, dY_k(u)\Big]$$

$$= \lambda_t(f) + E^*\Big(E^{Z(t)}\Big[\int_t^{t+s} f(Z(u))\, dY_k(u)\Big]\Big)$$

$$= \lambda_t(f) + E^*\Big[\int_0^{s} f(Z(u))\, dY_k(u)\Big]$$

$$= \lambda_t(f) + \lambda_s(f) \quad .$$

The antepenultimate equality follows from the Markov property of the (Z,Y) process and the fact that $\int_t^{t+s} f(Z(u))\, dY_k(u)$ belongs to the

where each integral is defined pathwise as a Lebesgue-Stieltjes integral. Line (79) now implies that $\nu_k(S_k) = \nu_k(S_k) = E^*(Y_k(1))$.

Q.E.D.

We will now prove Theorem 59. Take $f \in \underline{S}$ then Theorem 41 implies that

$$(80) \quad f(Z(t))-f(Z(0)) = \int_0^t Lf(Z(s))ds + \sum_{k=1}^{K} \int_0^t \frac{\partial f}{\partial x_k}(Z(s))\, dW_k(s)$$

$$+ \sum_{k=1}^{K+1} \int_0^t D_k f(Z(s))\, dY_k(s) .$$

Apply E^* to (80), and use the stationarity of Z and Fubini's Theorem to obtain

$$(81) \qquad 0 = t \int_S Lf(x)\, \Pi(dx) + \sum_{k=1}^{K+1} E^*\Big[\int_0^t D_k f(Z(s))\, dY_k(s)\Big] .$$

Apply Propositions 74 and 77 to (81) to obtain

$$(82) \qquad 0 = t \int_S Lf(x)\, \Pi(dx) + \sum_{k=1}^{K+1} t \int_{S_k} D_k f(x)\, \nu_k(dx) .$$

Dividing through by t we now obtain (60).

CHAPTER 6

A LIMIT THEOREM

In this chapter we consider a sequence of production networks indexed by $n = 1, 2, \ldots$. (Processes and quantities associated with the nth system will be indicated by a superscript n.) Each system in the sequence has the same number of work stations, denoted by $K+1$ as in Chapters 3 and 4, and has the same network structure, embodied in a fixed successor mapping σ as in Chapters 3 and 4. The sizes of the storage buffers and the stochastic character of the various potential output processes will be allowed to depend on n. Specifically, it will be assumed that storage buffer k is of size $b_k n^{1/2}$ in the nth system, where $b = (b_1, \ldots, b_K)$ is a fixed vector with strictly positive components.

6.1 The Main Result

Let (ψ, ϕ) be the reflection mapping on C^K defined in terms of σ as in Chapter 4. Recall that we write $\psi(x,b)$ and $\phi(x,b)$ when it is desirable to indicate explicitly the dependence of these maps on the capacity levels $b = (b_1, \ldots, b_K)$.

Let $(\Omega, \underline{B}, P)$ be a probability space upon which there are defined a sequence of potential output processes $\{\xi^{(n)}, n \geq 1\}$. Define $\{X^{(n)}, n \geq 1\}$ by

$$(1) \qquad X_k^{(n)} \equiv \xi_k^{(n)} - \xi_{\sigma(k)}^{(n)} \quad \text{for } k = 1, \ldots, K .$$

Alternately, we can represent $X^{(n)}$ by

(2) $$X^{(n)} \equiv F\xi^{(n)} \ ,$$

where F is a $K \times K{+}1$ matrix defined by

(3) $$f_{ij} \equiv \begin{cases} 1 & \text{for } j = i \\ -1 & \text{for } j = \sigma(i) \\ 0 & \text{otherwise} \end{cases} \quad .$$

Define $(Z^{(n)}, Y^{(n)})$ for $n \geq 1$ as follows:

(4) $$Z^{(n)}(t) \equiv n^{-1/2} \ \phi(X^{(n)},\ n^{1/2}\ b)\ (nt) \quad ,$$

and

(5) $$Y^{(n)}(t) \equiv n^{-1/2} \ \phi(X^{(n)},\ n^{1/2}\ b)\ (nt) \quad .$$

Referring to Chapters 3 and 4, we see that $\phi(X^{(n)}, n^{1/2}\ b)$ is the K-dimensional buffer contents process for a production network with potential output process $\xi^{(n)}$ and capacity $n^{1/2}b$. That is, $\phi(X^{(n)}, n^{1/2}b)$ is the contents process for our nth system, and $\phi(X^{(n)}, n^{1/2}\ b)$ is the associated loot potential output process. Then $Z^{(n)}$ and $Y^{(n)}$ are obtained from these, the processes of fundamental interest, by a rescaling of time and space.

(6) <u>Theorem</u>. Let $X^{(n)}$, $Z^{(n)}$, $Y^{(n)}$ be defined by (1), (4), (5) respectively. Suppose that

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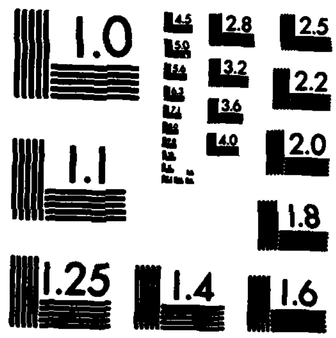

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$$(7) \qquad n^{-1/2} X^{(n)}(nt) \Rightarrow X \quad \text{as} \quad n \to \infty ,$$

where X is a Brownian motion with drift vector μ and covariance matrix A as in Chapter 5. Then

$$(8) \qquad (Z^{(n)}, Y^{(n)}) \Rightarrow (\phi(X,b), \psi(X,b)) \quad \text{as} \quad n \to \infty .$$

Proof. Lemma 4.75 implies that

$$(9) \qquad Z^{(n)}(t) = \phi(n^{-1/2} X^{(n)}(nt), b) ,$$

and

$$(10) \qquad Y^{(n)}(t) = \psi(n^{-1/2} X^{(n)}(nt), b) .$$

Theorem 4.69 implies that the mapping $x \to (\phi(x), \psi(x))$ is continuous from C^K to C^{2K+1}. Since $n^{-1/2} X^{(n)}(n\cdot)$ converges weakly to X, the continuous mapping theorem implies that

$$(11) \qquad (\phi(n^{-1/2} X^{(n)}(nt), b), \psi(n^{-1/2} X^{(n)}(nt), b)$$

$$\Rightarrow (\phi(X,b), \psi(X,b)) \quad \text{as} \quad n \to \infty .$$

Lines (9), (10) and (11) jointly imply (8). Q.E.D.

6.2 Two Applications

As specific examples of our general production network, we discussed in Chapter 3 a random walk model and a random environment

model. Theorem 12 below shows how the hypothesis of our main result (6) can be satisfied by a sequence of random walk models. The interpretation of this result in terms of random walk models will be clear from the notational parallels with Section 3.2.

(12) Theorem. Let $(\Omega, \underline{B}, P)$ be a probability space and suppose that for each $n \geq 1$ there exists $\{\zeta^{(n)}(m); m \geq 1\}$ a sequence of IID random $K+1$ vectors such that

(13) $$\lim_{n \to \infty} n^{1/2} E(F\zeta^{(n)}(1)) = \mu , \quad \text{where } \mu \in R^K ,$$

(14) $$\lim_{n \to \infty} \mathrm{Cov}(\zeta^{(n)}(1)) = D , \quad \text{where } D \text{ is a } K+1 \times K+1 \text{ positive definite matrix,}$$

and

(15) $$\lim_{n \to \infty} \int_{\{\|\zeta^{(n)}(1)\| \geq \varepsilon n^{1/2}\}} \|\zeta^{(n)}\|^2 \, dP = 0 \quad \text{for all } \varepsilon > 0 .$$

Define $\xi^{(n)}(t)$ as follows

(16) $$\xi^{(n)}(t) \equiv \sum_{m=1}^{[t]} \zeta^{(n)}(m) + (t-[t]) \, \zeta^{(n)}([t+1]) .$$

Define $X^{(n)}$ by

(17) $$X^{(n)} \equiv F\xi^{(n)} .$$

Let X be a K-dimensional Brownian motion starting at 0 with drift μ and covariance matrix $A = FDF'$. Finally define $(Z^{(n)}, Y^{(n)})$ by equations (4) and (5). Then

(18) $$(Z^{(n)}, Y^{(n)}) \Longrightarrow (\phi(X,b), \psi(X,b)) \qquad \text{as} \quad n \to \infty .$$

Proof. Lemma 20 (proved below) implies that

(19) $$\lim_{n \to \infty} n^{-1/2} X^{(n)}(n\cdot) \Longrightarrow X .$$

Line (19) and Theorem 6 jointly imply (18).

(20) Lemma. Let $(\Omega,\underline{B},P)$ be a probability space and suppose that for each $n \geq 1$ there exists $\{\eta^{(n)}(m);\ m \geq 1\}$ a sequence of IID random K-vectors such that

(21) $$\lim_{n \to \infty} \int_{\{\|\eta^{(n)}(1)\|>\varepsilon n^{1/2}\}} \|\eta^{(n)}(1)\|^2 \, dP = 0 , \qquad \text{for all} \quad \varepsilon > 0,$$

(22) $$\lim_{n \to \infty} \operatorname{Cov}(\eta^{(n)}(1) = A ,$$

and

(23) $$\lim_{n \to \infty} n^{1/2} E\eta^{(n)}(1) = \mu .$$

Define $X^{(n)}$ for $n \geq 1$ by

(24) $$X^{(n)} = \sum_{j=1}^{[t]} \eta^{(n)}(j) + (t - [t])\, \eta^{(n)}([t+1]) .$$

Then $n^{-1/2} X^{(n)}(n\cdot)$ converges weakly to X as $n \to \infty$, where X is a Brownian motion with drift vector μ and covariance matrix A.

Proof. Lemma 2.104 implies that it is sufficient to prove that $\{r_T(X^{(n)}), n \geq 1\}$ converges weakly to $r_T(X)$ for every $T > 0$ (see (2.103) for the definition of r_T). The weak convergence of $\{r_T(X^{(n)}), n \geq 1\}$ to $r_T(X)$ can be proved by imitating the proof of Theorem 4.1 of Parthasarathy [11]. Q.E.D.

Finally, we consider a sequence of random environment models which satisfy the hypothesis of our limit theorem (6). For simplicity, we suppose that the various systems in this sequence share a common environment process θ, but the work rates $r_k(m)$ for various states of the environment m change with sequence index n. The interpretation of the following in terms of random environment models will be clear from the notational parallels with Section 3.3.

(25) Theorem. Let $(\Omega,\underline{B},P)$ be a probability space and suppose that there is defined on $(\Omega,\underline{B},P)$ an ergodic Markov chain $\theta \equiv \{\theta(t), t \geq 0\}$ with state space $\{1, 2, \ldots, M\}$ and stationary distribution Π. For $n \geq 1$, let $r^{(n)}$ be a mapping from $\{1, 2, \ldots, M\}$ to R_+^{K+1}. Suppose that θ and $r^{(n)}$ jointly satisfy

(26) $$\theta(0) \sim \Pi ,$$

(27) $$E[Fr^{(n)}(\theta(0))]\, n^{1/2} \to \mu \qquad \text{as } n \to \infty, \text{ where } \mu \in R^K ,$$

and

(28) $$\lim_{n \to \infty} Fr^{(n)} = g , \qquad \text{where } \|g\| < \infty .$$

Define $w^{(n)}$ and $X^{(n)}$ respectively by

(29) $$w^{(n)}(s) = Fr^{(n)}(\theta^{(n)}(s)) ,$$

and

(30) $$X^{(n)}(t) = \int_0^t w^{(n)}(s)\, ds , \quad \text{for } t \geq 0 \text{ and } n \geq 1 .$$

As before define $(Z^{(n)}, Y^{(n)})$ by equations (4) and (5). Then $(Z^{(n)}, Y^{(n)})$ converges weakly to $(\phi(X,b), \phi(X,b))$, where X is a Brownian motion starting at 0 with drift μ and covariance matrix A defined by

(31) $$A \equiv 2 \int_0^\infty \mathrm{Cov}(g(\theta(0)), g(\theta(s)))\, ds .$$

Before we can begin the proof of Theorem 25, we need to consider some preliminary concepts.

Let $v = \{v(t), t \geq 0\}$ be a stochastic process from Ω to R^K. We define v to be <u>h-mixing</u> if

(32) $$\left| P(E_1 \cap E_2) - P(E_1) \cdot P(E_2) \right| \leq P(E_1)\, h(t) ,$$

holds whenever E_1 lies in the σ-field generated by $\{v(u); 0 \leq u \leq s\}$ and E_2 lies in the σ-field generated by $\{v(u); u \geq s+t\}$.

(33) <u>Lemma</u>. Suppose $\{v^{(n)}, n \geq 1\}$ is a sequence of strictly stationary stochastic processes on $(\Omega, \underline{B}, P)$ which jointly satisfy

(34) $v_s^{(n)}$ is a measurable function a.s. P,

(35) Each process $v^{(n)}$ is h-mixing,

(36) $$\int_0^\infty h^{1/2}(t)\, dt < \infty$$

(37) $$\lim_{n\to\infty} \int_{\{|v_0^{(n)}|\geq \varepsilon\, n^{1/2}\}} |v_0^{(n)}|\, dP = 0 \quad \text{for all} \quad \varepsilon > 0 ,$$

(38) $$\lim_{n\to\infty} \operatorname{Cov}(v_0^{(n)}, v_s^{(n)}) = \rho(s),$$

where ρ is a real valued matrix function, and

(39) $$\lim_{n\to\infty} n^{1/2}\, Ev_0^{(n)} = \mu \in R^K .$$

Define $X^{(n)}$ a random element of C^K by

(40) $$X^{(n)}(t) \equiv \int_0^t v_s^{(n)}\, ds \quad .$$

Then $n^{-1/2} X_n(n\cdot)$ converges weakly to X, where X is a Brownian motion with drift μ and covariance matrix $A = (a_{ij})$ defined by.

$$a_{ij} = 2 \int_0^\infty \rho_{ij}(s)\, ds \quad . \tag{41}$$

The proof of Lemma 33 is almost identical to that of Theorem 20.1 of Billingsley [1], and therefore we omit it. Q.E.D.

We will now prove Theorem 25. Since θ is an ergodic Markov chain, it is possible to show that there exists $h(t) \equiv ar^t$ (where $a > 0$ and $r < 1$) such that θ is h-mixing. Consequently $w^{(n)}$ must also be h-mixing. Clearly h satisfies condition (36) of Lemma 33. Observe that $\sup_{n \geq 1} \|Fr^{(n)}\| < \infty$ and therefore $w^{(n)}$ satisfies (27). Line (28) guarantees that

$$\lim_{n \to \infty} \mathrm{Cov}(w^{(n)}(0), w^{(n)}(s)) = \mathrm{Cov}(g(\theta(0)), g(\theta(s))) \; . \tag{42}$$

Finally observe that $w^{(n)}$ is measurable almost surely (P) because θ is measurable almost surely (P). Therefore $\{w^{(n)}, n \geq 1\}$ satisfies the hypotheses of Lemma 33 and thus

$$\lim_{n \to \infty} n^{-1/2} X^{(n)}(n\cdot) \Longrightarrow X \; , \tag{43}$$

where X is a Brownian motion starting at 0 with drift μ and covariance A. Theorem 6 now implies the desired result.

(44) <u>Remark</u>. Condition 26 may be omitted from Theorem 25 without changing its conclusions.

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ABSTRACT: A PRODUCTION NETWORK MODEL AND ITS DIFFUSION APPROXIMATION
by Michael Louis Wenocur Report No. 207

This report develops and analyzes a general stochastic model of a production system. The model is closely related to Harrison's [5] assembly-like queueing network, the principal difference being that here we assume all storage buffers have finite capacity. Our attention is focused on a vector stochastic process Z whose components are the contents of the various storage buffers (as functions of time). The principal result is a weak convergence theorem of the type developed by Iglehart and Whitt [7] for queues in heavy traffic. This limit theorem shows that, with large buffers and balanced loading of the system's work stations (see below), a properly normalized version of the storage process Z can be well approximated by a certain vector diffusion process Z^*. We construct Z^* by applying a particular (and rather complicated) reflection mapping to multidimensional Brownian motion. Various properties of the limiting diffusion Z^* are developed, but these provide only a modest beginning for the analytical theory that must be developed before our limit theorem can lead to practically useful approximation procedures.